Question: Consider the *double* delta-function potential

$$V(x) = -\alpha\left(\delta(x+a) + \delta(x-a)\right) \tag{1}$$

where α and a are a positive constants. How many bound states does it possess? Find the allowed energies, for $\alpha = \hbar^2/ma$ and for $\alpha = \hbar^2/4ma$, and sketch the wave function.

My solution
Part I

I know how to solve the problem for a single delta-function potential so I use that solution as a guide. For bound states, the total energy $E < 0$. The TISE for the region $x < -a$ where $V(x) = 0$, can be written as

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = k^2\psi \tag{2}$$

where

$$k = \frac{\sqrt{-2mE}}{\hbar} \tag{3}$$

The general solution to (2) can be written as,

$$\psi(x) = Ae^{-kx} + Be^{kx} \tag{4}$$

We must have $A = 0$ or else $\psi(x)$ blows up as $x \to -\infty$. Hence (4) reduces to

$$\psi(x) = Be^{kx} \qquad \forall \qquad x < -a \tag{5}$$

The TISE has the same form for $-a < x < a$ but the solution is a sum of (surviving) exponentials,

$$\psi(x) = Ce^{-kx} + De^{kx} \qquad \forall \qquad -a < x < a \tag{6}$$

For $x > a$, the TISE again has the same form. The general solution is

$$\psi(x) = Fe^{-kx} + Ge^{kx} \tag{7}$$

We must have $G = 0$ or else $\psi(x)$ blows up as $x \to \infty$. Hence (8) reduces to,

$$\psi(x) = Fe^{-kx} \qquad \forall \qquad x > a \tag{8}$$

For $\psi(x)$ to be continuous everywhere, we must have

$$Be^{-ka} = Ce^{ka} + De^{-ka} \tag{9}$$

$$Ce^{-ka} + De^{ka} = Fe^{-ka} \tag{10}$$

PART 2

The 1-D TISE with the double-delta potential can be written as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\left[\delta(x+a) + \delta(x-a)\right]\psi = E\psi \tag{11}$$

Integrating both sides with respect to x over the interval $-a-\epsilon$ to $-a+\epsilon$, taking the limit $\epsilon \to 0$ and transposing terms, we get

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=-a} \equiv \lim_{\epsilon\to 0}\left[\left[\frac{d\psi}{dx}\right]_{-a+\epsilon} - \left[\frac{d\psi}{dx}\right]_{-a-\epsilon}\right] = -\frac{2m\alpha}{\hbar^2}\psi(-a) \tag{12}$$

Similarly,

$$\Delta\left(\frac{d\psi}{dx}\right)_{x=a} \equiv \lim_{\epsilon\to 0}\left[\left[\frac{d\psi}{dx}\right]_{a+\epsilon} - \left[\frac{d\psi}{dx}\right]_{a-\epsilon}\right] = -\frac{2m\alpha}{\hbar^2}\psi(a) \tag{13}$$

Now, using the solutions (5), (6), (8) to evaluate $d\psi/dx$ in different intervals, and equations (12) and (13), we get

$$-2kCe^{ka} = -\frac{2m\alpha}{\hbar^2}Be^{-ka} \tag{14}$$

$$-2kBe^{ka} = -\frac{2m\alpha}{\hbar^2}De^{-ka} \tag{15}$$

I think the problem reduces to stitching the wavefunction using equations (9) and (10) and equations (14) and (15) together to somehow evaluate k (in terms m, α, $\hbar$ and a) in order that one may use equation (3) to get an expression for E. The problem is with the algebraic manipulations assuming that the algebra so far is correct. If my guess is right, we are dealing with nonlinear equations (k appears both as a coefficient and as an exponent).

PS: I get $CD = B^2$ from equations (14) and (15).