The compass sitting ontop of another compass with a wheel drawing circles problem (aka the roundabout on top of a piece of paper problem).

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To find the area enclosed by a closed contour in polar coordinates (assuming that the contour also encloses the origin) we say that

$$A = \int_0^{2\pi} d\phi \int_0^{r(\phi)} r dr = \frac{1}{2}\int_0^{2\pi} d\phi r^2 \left(\phi\right),$$

where $r\left(\phi\right)$ describes the radial distance to the contour for a given angle ϕ (where $\phi = 0$ takes you positively along the x-axis). Now if the curve is given by some parameter t then this becomes

$$A = \frac{1}{2}\int_{t_0}^{t_1} dt \frac{d\phi\left(t\right)}{dt} r^2\left(t\right).$$

The limits in the above integral are chosen such that $\phi\left(t_0\right) = 0$ and $\phi\left(t_1\right) = 2\pi$. So if you're given some curve $v\left(t\right) = \left(x\left(t\right), y\left(t\right)\right)$, then the task is to find its representation in polar coordinates $v\left(t\right) = \left(r\left(t\right), \phi\left(t\right)\right)$. You can then use these in the expression for A.

The problem is easier if you say

$$z\left(t\right) = x\left(t\right) + iy\left(t\right) = r\left(t\right) e^{i\phi(t)}$$

then in the complex plane your contour is described by $z\left(t\right)$, with some radial ordinate $r\left(t\right)$ and an angular ordinate $\phi\left(t\right)$. Note that if you take real, or imaginary parts of $z\left(t\right)$ you will recover $x\left(t\right)$ or $y\left(t\right)$. Given your two compasses, $z\left(t\right)$ will look like:

$$z\left(t\right) = R_1 e^{i\omega t} + R_2 e^{2i\omega t}.$$

(where the position of compass 1 is $R_1 e^{i\omega t}$ etc).

0.1 Skip this bit if you can't be bothered reading it.

For this problem to make sense, we have to demand that the every point on the curve is determined by a unique value of t (up to $2\pi/\omega$, since $z(t)$ is periodic on this range), otherwise we will have crossing points and this will lead to having the contour enclosing a part of itself, or two connected areas or somesuch (probably the latter case is soluble, but the former doesn't make much sense). So, if we look at z for some value of t and another value t', if these z are the same then

$$\frac{z(t)}{z(t')} = 1$$

and the problem is non doable (or can't be botheredable), and for $z(t') = 0$, then we require $z(t) = 0$. So

$$\frac{z(t)}{z(t')} = \frac{R_1 e^{i\omega t} + R_2 e^{2i\omega t}}{R_1 e^{i\omega t'} + R_2 e^{2i\omega t'}} = 1.$$

Obviously $t = t'$ is a solution, otherwise,

$$R_1 \left(e^{i\omega t} - e^{i\omega t'}\right) = R_2 \left(e^{2i\omega t'} - e^{2i\omega t}\right),$$

and so

$$\begin{aligned}
\frac{R_1}{R_2} &= \frac{\left(e^{2i\omega t'} - e^{2i\omega t}\right)}{\left(e^{i\omega t} - e^{i\omega t'}\right)} = \frac{\left(e^{i\omega t'} - e^{i\omega t}\right)\left(e^{i\omega t'} + e^{i\omega t}\right)}{\left(e^{i\omega t} - e^{i\omega t'}\right)}, \\
\frac{R_1}{R_2} &= -\left[\cos(\omega t') + \cos(\omega t)\right] - i\left[\sin(\omega t') + \sin(\omega t)\right].
\end{aligned}$$

Now since the LHS is real (R_1 and R_2 are just radii) then we require that

$$\left[\sin(\omega t') + \sin(\omega t)\right] = 0$$

and

$$\frac{R_1}{R_2} = -\left[\cos(\omega t') + \cos(\omega t)\right].$$

Since the RHS is always < 2 (given that $t' \neq t$), if we choose the ratio

$$\frac{R_1}{R_2} > 2, \qquad \text{which is} \qquad R_2 < \frac{R_1}{2}$$

then each point on the contour will be described by a unique value of t. Anyway, carrying on...

0.2 Start reading again

So now if you want to find the distance squared from the origin to any point on $z(t)$, this is just the modulus squared

$$r^2(t) = z(t)\, z^*(t) = |z(t)|^2.$$

So for the compasses:

$$r^2(t) = R_1^2 + R_2^2 + 2R_1R_2\cos(\omega t).$$

If we want to find the time derivative of $\phi(t)$, we start by finding

$$e^{i\phi(t)} = \frac{z(t)}{r(t)},$$

and therefore

$$e^{2i\phi(t)} = \frac{z^2(t)}{r^2(t)} = \frac{z^2(t)}{z(t)\,z^*(t)} = \frac{z(t)}{z^*(t)}.$$

So

$$2i\frac{d}{dt}\phi(t) = \frac{1}{e^{2i\phi(t)}}\frac{d}{dt}e^{2i\phi(t)} = \left(\frac{z^*(t)}{z(t)}\right)\frac{d}{dt}\left(\frac{z(t)}{z^*(t)}\right) = \frac{1}{z(t)}\frac{d}{dt}z(t) - \frac{1}{z^*(t)}\frac{d}{dt}z^*(t).$$

Looking at the explicit expressions for $z(t)$

$$\frac{d}{dt}z(t) = R_1e^{i\omega t} + R_2e^{2i\omega t} = i\omega R_1e^{i\omega t} + 2i\omega R_2e^{2i\omega t} = i\omega z(t) + i\omega R_2e^{2i\omega t},$$
$$\frac{d}{dt}z^*(t) = R_1e^{-i\omega t} + R_2e^{-2i\omega t} = -i\omega R_1e^{-i\omega t} - 2i\omega R_2e^{-2i\omega t} = -i\omega z^*(t) - i\omega R_2e^{-2i\omega t}.$$

Hence

$$\begin{aligned} 2i\frac{d}{dt}\phi(t) &= \frac{i\omega z(t) + i\omega R_2e^{2i\omega t}}{z(t)} - \frac{\left(-i\omega z^*(t) - i\omega R_2e^{-2i\omega t}\right)}{z^*(t)} \\ &= 2i\omega + 2i\omega R_2\,\mathrm{Re}\left(\frac{e^{2i\omega t}}{z(t)}\right). \end{aligned}$$

We can simplify this further

$$\begin{aligned} \frac{d}{dt}\phi(t) &= \omega + \frac{\omega R_2}{|z(t)|^2}\,\mathrm{Re}\left(z^*(t)\,e^{2i\omega t}\right) \\ &= \omega + \frac{\omega R_2}{r^2(t)}\,\mathrm{Re}\left(\left[R_1e^{-i\omega t} + R_2e^{-2i\omega t}\right]e^{2i\omega t}\right) \\ &= \omega + \frac{\omega R_2R_1}{r^2(t)}\cos(\omega t) + \frac{\omega R_2^2}{r^2(t)}. \end{aligned}$$

(You can also get the same expression for the maximum allowed R_2 by demanding that $\dot{\phi}(t) > 0$ here). Putting $r^2(t)$ and $\dot{\phi}(t)$ into the expression for A gives

$$\begin{aligned} A &= \frac{1}{2}\int_{t_0}^{t_1} dt\left[\omega + \frac{\omega R_2R_1}{r^2(t)}\cos(\omega t) + \frac{\omega R_2^2}{r^2(t)}\right]r^2(t) \\ &= \frac{\omega}{2}\int_{t_0}^{t_1} dt\left[r^2(t) + R_2R_1\cos(\omega t) + R_2^2\right] \\ &= \frac{\omega}{2}\int_{t_0}^{t_1} dt\left[R_1^2 + 2R_2^2 + 3R_1R_2\cos(\omega t)\right] \end{aligned}$$

Next we have to choose limits of integration. The lower limit corresponds to $t = 0$, and since $z\left(t\right)$ is periodic on $2\pi/\omega$, this becomes the upper limit,

$$A = \frac{\omega}{2} \int_0^{2\pi/\omega} dt \left[R_1^2 + 2R_2^2 + 3R_1 R_2 \cos\left(\omega t\right)\right] = \pi R_1^2 + 2\pi R_2^2.$$

0.3 Conclusion

I've no idea if the above result is correct. If someone could check it Duhoc would be very greatful. If it *is* correct, then if someone could offer a hand-waving interpretation of the second term then that would be nice.