A "Geometric" Derivation Of The Lorentz Transformations

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The following property of the geometric mean can be applied to the Galilean coordinates of the wavefronts from stationary and moving sources. This will make the wavefronts coincident and the speeds of the wavefronts will be c. However, the clock at the coincident wavefronts will no longer run at the same rate as the universal clock τ of Galilean space-time.

The quantity $\sqrt{y/x}$ and it's reciprocal, $\sqrt{x/y}$, map between x, y, and the geometric mean $\sqrt{xy}$. This is analogous to $(y-x)/2$ and it's additive inverse mapping between x, y, and the arithmetic mean $(x+y)/2$.

$$x\sqrt{\frac{y}{x}} = \sqrt{x^2\frac{y}{x}} = \sqrt{xy}$$

$$\sqrt{xy}\sqrt{\frac{y}{x}} = \sqrt{xy\frac{y}{x}} = \sqrt{y^2} = y$$

The quantity $\sqrt{y/x}$ maps from x to the geometric mean of x and y to y.

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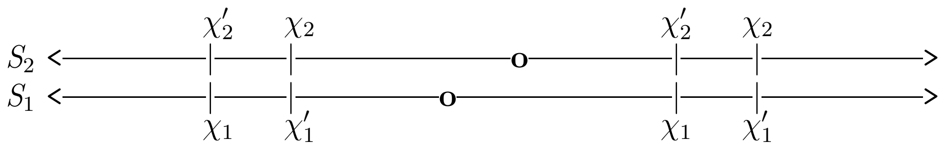
The quantity $\sqrt{x/y}$ maps from y to the geometric mean of x and y to x.

This property will now be applied to the Galilean coordinates of light pulses that originated from the origins of two frames of reference moving with a relative speed of v. The rate of the clock t relative to τ that will make the wavefronts from both origins coincident with constant speed c will be derived. The relative rates of the clocks in the two frames of reference at the coincident wavefronts will be derived from these results. The Lorentz transforms for t and x will then be derived from this relationship.

In the following diagram, S_2 is moving with speed v in the positive x direction with respect to S_1, and their origins are designated by o.



χ_i and χ_i' are the Galilean locations of the wavefronts from the stationary and moving sources, respectively.

The wavefronts were produced at the origins of S_1 and S_2 when the origins were coincident, and the clocks at both origins read 0.. That is, when $x_1 = x_2 = 0$ and $t_1 = t_2 = 0$.

It is assumed that the clocks in both frames are synchronized with the universal Galilean time τ.

Note that the initial conditions obscure the fact that the derived relationship between t_1 and t_2 is for their relative rates, not their actual values.

For S_1 at the right wavefronts:

1) $\chi_1 = c\tau$
2) $\chi_1' = (c+v)\tau$

From 1 and 2:

$\sqrt{\chi_1\chi_1'} = \sqrt{c\tau(c+v)\tau}$, so $\sqrt{\dfrac{\chi_1'}{\chi_1}} = \sqrt{\dfrac{(c+v)\tau}{c\tau}} = \sqrt{\dfrac{c+v}{c}}$

Let:

3) $x_1 = \sqrt{\chi_1\chi_1'}$

4) $k_1 = \sqrt{\dfrac{c+v}{c}}$

5) $t_1 = k_1\tau$.

From 3, 4, and the previously described property of the geometric mean:

6) $x_1 = k_1\chi_1$ and

7) $x_1 = \chi_1'/k_1$.

Solving for x_1/t_1 from the stationary source using 5 and 6, then 1:

$\dfrac{x_1}{t_1} = \dfrac{k_1\chi_1}{k_1\tau} = \dfrac{\chi_1}{\tau} = c$

Doing the same for the moving source using 5 and 7, then 2, then 4:

$\dfrac{x_1}{t_1} = \dfrac{\chi_1'/k_1}{k_1\tau} = \dfrac{1}{k_1^2}\dfrac{\chi_1'}{\tau} = \dfrac{1}{k_1^2}(c+v) = \left(\sqrt{\dfrac{c}{c+v}}\right)^2 (c+v) = c$

Clearly, the speed of light is constant for the stationary source at the origin of S_1 and for the moving source at the origin of S_2. Also, the wavefronts from both sources are coincident at x_1, the geometric mean of their Galilean coordinates, χ_1 and χ_1' .

Substituting 6 into 7 and clearing the denominator:
8) $t_1 = \tau\sqrt{1 + v/c}$

For S_2 at the right wavefronts:

9) $\chi_2 = c\tau$
10) $\chi_2' = (c - v)\tau$

From 9 and 10:

$\sqrt{\chi_2\chi_2'} = \sqrt{c\tau(c - v)\tau}$, so $\sqrt{\dfrac{\chi_2'}{\chi_2}} = \sqrt{\dfrac{(c - v)\tau}{c\tau}} = \sqrt{\dfrac{c - v}{c}}$

Let:
11) $x_2 = \sqrt{\chi_2\chi_2'}$
12) $k_2 = \sqrt{\dfrac{c - v}{c}}$
13) $t_2 = k_2\tau$.

From 11, 12, and the previously described property of the geometric mean:
14) $x_2 = k_2\chi_2$ and
15) $x_2 = \chi_2'/k_2$.

Solving for x_2/t_2 from the stationary source using 13 and 14, then 9:

$\dfrac{x_2}{t_2} = \dfrac{k_2\chi_2}{k_2\tau} = \dfrac{\chi_2}{\tau} = c$

Doing the same for the moving source using 13 and 15, then 10, then 12:

$\dfrac{x_2}{t_2} = \dfrac{\chi_2'/k_2}{k_2\tau} = \dfrac{1}{k_2^2}\dfrac{\chi_2'}{\tau} = \dfrac{1}{k_2^2}(c - v) = \left(\sqrt{\dfrac{c}{c - v}}\right)^2 (c - v) = c$

Again, the speed of light is constant for the stationary source at the origin of S_2 and for the moving source at the origin of S_1. Also, the wavefronts from both sources are coincident at x_2, the geometric mean of their Galilean coordinates, χ_2 and χ_2'.

Substituting 12 into 13 and clearing the denominator:
16) $t_2 = \tau\sqrt{1 - v/c}$

Dividing 16 by 8 and cancelling the common term τ:

17a) $\dfrac{t_2}{t_1} = \sqrt{\dfrac{1 - v/c}{1 + v/c}}$

$$\frac{t_2}{t_1} = \sqrt{\frac{(1-v/c)(1-v/c)}{(1-v/c)(1+v/c)}} = \sqrt{\frac{(1-v/c)^2}{(1-v/c)(1+v/c)}} = \frac{1-v/c}{\sqrt{1-v^2/c^2}}$$

17b) $t_2 = t_1 \dfrac{1-v/c}{\sqrt{1-v^2/c^2}}$

Since $x_1 = ct_1$ at the coincident wavefronts, x_1/c can be substituted for t_1:

$$t_2 = \frac{t_1 - t_1 v/c}{\sqrt{1-v^2/c^2}} = \frac{t_1 - (x_1/c)(v/c)}{\sqrt{1-v^2/c^2}}$$

18) $t_2 = \dfrac{t_1 - x_1 v/c^2}{\sqrt{1-v^2/c^2}}$

The transform for the x coordinate can be derived from 17b by multiplying both sides by c, rearranging, and making the appropriate substitutions of x_i for ct_i.

$$ct_2 = ct_1 \frac{1-v/c}{\sqrt{1-v^2/c^2}} = \frac{ct_1 - ct_1 v/c}{\sqrt{1-v^2/c^2}} = \frac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}$$

19) $x_2 = \dfrac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}$

Equations 18 and 19 are the Lorentz transforms for t_2 and x_2 respectively.

The inverse transforms are derived by solving 17a for t_1 and x_1 in the same manner as for t_2 and x_2:

$$\frac{t_1}{t_2} = \sqrt{\frac{1+v/c}{1-v/c}}$$

$$\frac{t_1}{t_2} = \sqrt{\frac{(1+v/c)(1+v/c)}{(1+v/c)(1-v/c)}} = \sqrt{\frac{(1+v/c)^2}{(1+v/c)(1-v/c)}} = \frac{1+v/c}{\sqrt{1-v^2/c^2}}$$

$$t_1 = t_2 \frac{1+v/c}{\sqrt{1-v^2/c^2}}$$

$$t_1 = \frac{t_2 + t_2 v/c}{\sqrt{1-v^2/c^2}} = \frac{t_2 + (x_2/c)(v/c)}{\sqrt{1-v^2/c^2}}$$

20) $t_1 = \dfrac{t_2 + x_2 v/c^2}{\sqrt{1-v^2/c^2}}$

$$ct_1 = ct_2 \frac{1+v/c}{\sqrt{1-v^2/c^2}} = \frac{ct_2 + ct_1 v/c}{\sqrt{1-v^2/c^2}} = \frac{x_2 + vt_2}{\sqrt{1-v^2/c^2}}$$

21) $x_1 = \dfrac{x_2 + vt_2}{\sqrt{1-v^2/c^2}}$

Equations 20 and 21 are the Lorentz transforms for t_1 and x_1 respectively.

For S_1 and S_2 at the left wavefronts:

22) $\chi_1 = -c\tau$
23) $\chi_1' = -(c-v)\tau$

24) $\chi_2 = -c\tau$
25) $\chi_2' = -(c+v)\tau$

Notice that in addition to $\chi < 0$, $c-v$ and $c+v$ are transposed between S_1 and S_2 compared to the right wavefronts. The negative signs will cancel for $\sqrt{\chi_i \chi_i'}$ and $\sqrt{\chi_i'/\chi_i}$, so x_i and k_i are real numbers. However, comparing 23 to 2 and 25 to 10, it is clear by inspection that at the left wavefronts:
$t_1 = \tau\sqrt{1-v/c}$
$t_2 = \tau\sqrt{1+v/c}$, so that

26a) $\dfrac{t_2}{t_1} = \sqrt{\dfrac{1+v/c}{1-v/c}}$

$$\frac{t_2}{t_1} = \sqrt{\frac{(1+v/c)(1+v/c)}{(1+v/c)(1-v/c)}} = \sqrt{\frac{(1+v/c)^2}{(1+v/c)(1-v/c)}} = \frac{1+v/c}{\sqrt{1-v^2/c^2}}$$

26b) $t_2 = t_1 \dfrac{1+v/c}{\sqrt{1-v^2/c^2}}$

Since $x_1 = -ct_1$ at the coincident wavefronts, $-x_1/c$ can be substituted for t_1:

$$t_2 = \frac{t_1 + t_1 v/c}{\sqrt{1-v^2/c^2}} = \frac{t_1 + (-x_1/c)(v/c)}{\sqrt{1-v^2/c^2}}$$

27) $t_2 = \dfrac{t_1 - x_1 v/c^2}{\sqrt{1-v^2/c^2}}$

The transform for the x coordinate can be derived from 26b by multiplying both sides by -c, rearranging, and making the appropriate substitutions of x_i for $-ct_i$.

$$-ct_2 = -ct_1 \frac{1+v/c}{\sqrt{1-v^2/c^2}} = \frac{-ct_1 + (-ct_1)v/c}{\sqrt{1-v^2/c^2}} = \frac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}$$

28) $x_2 = \dfrac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}$

Equations are 27 and 28 identical to 18 and 19, as expected. Inverse transforms identical to 20 and 21 can readily be derived at the left wavefronts.

Equations 17a and 26a can be rewritten as one equation by a change of notation:

$$\frac{t_{same}}{t_{opposite}} = \sqrt{\frac{1 - v/c}{1 + v/c}}$$

So, if an observer could ride the coincident wavefronts from stationary and moving sources, he would say that time appears to pass much more slowly in the frame of reference moving in the same direction as the wavefronts relative to the frame of reference moving in the opposite direction of the wavefronts.