8320 day 16, Monday Feb. 15. Monomial singularities : $\{y^n = x^m\}$

We have shown how to desingularize a reduced plane curve, replacing a finite number of singular points by smooth points. Our general analysis gives at least a topological local structure theorem for reduced plane curves near every point. Recall a curve $\{F=0\}$ is called reduced if the polynomial F factors into distinct irreducible factors. Notice that the zero locus of a polynomial $\prod f_j^{r_j}$, where the f_j are distinct irreducible factors, equals the zero locus of the reduced polynomial $\prod f_j$, so if we are interested only in the point set defined by a polynomial we could always assume our polynomial is reduced. In more advanced contexts one does want to distinguish reduced from non reduced curves, since they have different deformations. I.e. the curve defined by y^3 deforms to a smooth plane cubic of genus one, while that defined by y deforms only to other lines of genus zero. So non reduced curves are important in the study of curves moving in families. But for now we restrict to reduced ones. In some sense this restriction is impossible to maintain though, since we often analyze our curves by looking first at the lowest degree term, and this term may not be reduced. When the lowest degree term, called the tangent cone, is reduced, then Hensel's lemma tells us the local analytic structure of our curve. Today we will learn to analyze also an important class of easy singularities whose tangent cones are not reduced, but whose local equations are simple enough to analyze fully anyway, "monomial" singularities.

First of all we know there are no isolated points on a reduced (complex) curve, and near a smooth, i.e. "non singular", point the implicit function theorem implies the curve has a neighborhood isomorphic to a disc. In the neighborhood of a singular point the curve can be extremely complicated, but it simplifies upon removal of the singularity. I.e. a singular point on a reduced curve has a small punctured neighborhood which is a finite covering space of a punctured disc, and this implies the punctured neighborhood itself is holomorphically isomorphic to a disjoint union of a finite number of punctured discs. Filling in each of these punctured discs by a disc, and doing this at every singular point, we obtain the Riemann surface associated to this curve. Thus every singular point has a neighborhood which is homeomorphic to a "one point union" of a finite number of discs, i.e. a finite number of discs which are disjoint except for all having the same center point.

This description is simple enough theoretically, but in practice it can be challenging to compute how many smooth points are needed to replace a given singular point, i.e. how many "branches" there are at the singular point. This number equals the number of connected components of a small punctured neighborhood of the singularity, but it is not always obvious how many there are. One needs to know this in order to compute the genus of the Riemann surface. E.g. if one tries to compute the genus by Hurwitz' formula one needs the total ramification index of the projection from the curve to the x axis, and this depends on the number of analytic branches at the point. If one tries to compute the genus of the Riemann surface by comparing it to that of a nearby smoothing using the Milnor number, one needs again to know the number of analytic branches. I.e. if the Milnor number of the singularity is μ and the number of analytic branches is r, one obtains a local topological smoothing by plumbing in a handlebody which is a connected 2-manifold of genus $(1/2)(\mu + 1 - r)$ with r discs removed, so the change in the local topology from the singular curve to the smoothing depends on both μ and r.

The simplest singularity, a local analytic model of a node

A nice class of examples for which one can compute all these numbers is the class of monomial singularities, which have local analytic equation of form y^n = x^m, in some local analytic coordinates, where n,m ≥ 2. The simplest case is n = m = 2, called a node, or ordinary double point (odp), with analytic equation y^2 = x^2. Here the local analytic equation factors as y^2-x^2 = (y-x)(y+x), these two factors each define a smooth local analytic branch of the curve near the singular point and the two branches are transverse. Thus the Riemann surface is obtained by separating these two branches and filling in the two holes in the two punctured discs comprising a small punctured neighborhood of the node. In this case the local equation equals the equation of the tangent cone and it is reduced, i.e. the two linear factors (y-x), (y+x) define distinct lines.

Since the two branches are smooth and transverse, the Milnor number here is 1, so a local smoothing is obtained by plumbing in a sphere with two discs removed. Notice that two disjoint discs have as boundary two disjoint circles, and that a sphere with two discs removed also has that same boundary. Thus when we remove a small neighborhood of the singularity we leave a hole whose boundary is two disjoint circles and these are two different ways to plug those holes. Putting in two disjoint discs yields the local Riemann surface, while putting in one sphere with two holes yields a local topological smoothing. One difference is the Riemann surface separates the distinct analytic branches, while the smoothing connects them up. Also the Riemann surface construction yields a complex analytic space while the smoothing construction gives only a topological model of a nearby complex plane curve. Every singularity with two local analytic branches is smoothed by plumbing in some connected 2 manifold with two disjoint circles as boundary, but knowing what this manifold is depends on knowing the Milnor number.

Recognizing a node from its global polynomial equation

We also want to be able to recognize a node given by a more complicated equation, since we usually have a global polynomial equation for our curve and not a local analytic equation. This too is easy by "Hensel's lemma" as follows. We can change coordinates linearly so that our singularity is at (0,0) in affine coordinates. Then the equation has form f(n) + f(n+1) + f(n+2) +..... where f(j) is a homogeneous polynomial of degree j in (x,y). Then (0,0) is a node if and only if the lowest order term f(n) appearing here has degree 2, and moreover f(2) factors into two distinct linear factors. I.e. if the lowest order homogenous term f(2) factors into two distinct linear factors, then the whole equation factors locally near (0,0) into two convergent power series whose linear terms are the two linear factors of the quadratic term.

This is easy to see at least formally. I.e. it is easy to write down two power series inductively, whose product equals the given equation, but it is a little more work to show those power series converge on some neighborhood of (0,0). We just do an example that illustrates the general case. Take the curve to be xy + x^3 + y^3 = 0. We want to factor it as xy + x^3 + y^3 = (x+g2+g3+g4+...)(y+h2+h3+h4+...), where gj and hj are homogeneous polynomials in (x,y) of degree j. This requires xh2 + yg2 = x^3 + y^3, which is readily solved by $h2 = x^2$, $g2 = y^2$. Next we need $xh3 + yg3 + g2h2 = 0$, and since we have already found g2,h2, this says $xh3 + yg3 + y^2x^2 = 0$, or $xh3 + yg3 = -x^2y^2$. This too is easily solved by $h3 = -xy^2$, $g3 = 0$, or several other choices. Next we want $xh4 + yg4 + g2h3 + g3h2 = 0$, or $xh4 + yg4 = - g2h3 - g3h2 = xy^4$, again easily solved. At each stage we only need to solve an equation of form $x\, hj + y\, gj = f(j+1)$, where $f(j+1)$ is homogeneous of degree j+1 in x,y. Since every term of $f(j+1)$ is divisible by either x or y, thus can always be done. The puzzle that there are many different solutions even though the power series ring is a "unique factorization domain" is explained when we recall that "uniqueness" of factorization is only true up to multiplication by units. I.e. there are infinitely many units in the ring of power series, since every power series with non zero constant term is invertible, so indeed all these different pairs of power series solutions for the factors must be associates.

As to convergence of these power series in the usual complex topology, note that in this particular example at least, every coefficient of every monomial in each factor above is 1, so these monomials are all bounded by 1 in the polydisc $|x|<1$, $|y| < 1$, hence the series converges there. (Recall that a power series converges in any polydisc where all the monomials are uniformly bounded.)

Ordinary n fold points are also determined by the lowest order local term

Apparently there is also a version of Hensel's lemma for singularities of higher multiplicity, as long as the lowest order term factors into distinct linear factors, i.e. when that homogeneous term is reduced, but I have not found an explicit reference. Thus if the curve has equation $f = f(d) + f(d+1) +.....$, where the homogeneous polynomial $f(d)$ factors into d distinct linear factors, then it seems f itself factors locally into d convergent power series each defining a smooth branch of the curve at (0,0) with tangent line defined by one of the linear factors of $f(d)$.

For example if the lowest order term of f is a monomial singularity of form $f = y^n-x^n$, which factors as $\prod(y - \mu j.x)$ where the μj are the distinct nth roots of 1, then also the equation itself factors locally into n power series each defining a smooth local branch with tangent line defined by one of the linear factors $(y - \mu j.x)$.

For this curve, and for all ordinary n fold points, the Milnor number $\mu = \dim k[[x,y]]/(fx,fy) = \dim k[[x,y]]/(x^{(n-1)},y^{(n-1)}) = (n-1)^2$. Since there are n local analytic branches, the Riemann surface is formed just by separating those n branches and filling in the n holes by n smooth points, one on each branch. On the other hand, if the curve is smoothed locally, the handlebody that is plumbed in is a compact connected surface of genus $= (1/2)(\mu+1-n)$ from which n discs have been removed. The genus of this handlebody, before removing the n discs, is thus $(1/2)(n-1)(n-2)$. If the curve is globally irreducible, this surgery increases the genus (from that of the Riemann surface to that of the smoothing) by $(1/2)(\mu+n-1) = (1/2)(n)(n-1)$.

For example, an ordinary triple point with local analytic equation $y^3-x^3 = 0$, may be thought of as 3 odp's coming together, hence lowers the genus of the Riemann surface by three, compared with a nearby local smoothing. An ordinary quadruple point with local analytic equation $y^4-x^4 = 0$ similarly lowers the genus by 6. In general an ordinary n fold point may be thought of as a coalescence of $(1/2)(n)(n-1)$ ordinary double points, and this is another way to see that it causes the genus of the Riemann surface of an irreducible curve to go down by $(1/2)(n)(n-1)$ compared to the genus of a nearby local smoothing.

If not reduced, the lowest order term does not fully determine the branches

From what I have read of related similar Hensel type results, I would guess that even if the lowest degree term factors as $f(d) = \prod Lj^{rj}$, where the Lj are distinct linear factors, but they may occur to higher multiplicities rj, then the full equation f also factors locally into at least as many power series factors as there are distinct linear factors Lj. For example, if the lowest term of f is x^3y2, then f should factor locally into at least two distinct power series, one with lowest term x^3 and the other with lowest term y^2, but these two power series may or may not factor further. Hence this curve has at least two distinct analytic branches at (0,0), but there may be more.

[This can be proved geometrically by invoking some more powerful theorems from analytic geometry. I.e. by our general local theory, every local analytic branch at this singularity can be parametrized by an analytic map from a disc. The image of such a parametrization, even if singular, has a unique tangent line, so there must be at least as many local parametrizations as there are lines in the tangent cone. If we know also that the image of every local parametrization has an analytic equation as a power series, a corollary of a "proper mapping" theorem, we would know these power series factors of our original equation must exist.]

In fact an equation whose lowest term is y^n, may be locally irreducible, or may factor into any number of local analytic factors between 1 and n. For example (y^4-x^8) = (y-x^2)(y+x^2)(y-ix^2)(y+ix^2) has 4 smooth branches; y^4 –y^2x^3- y^2x^4 +x^7 = (y^2-x^3)(y-x^2)(y+x^2) has 3 branches, two of which are smooth; y^4 –yx^5 –y^3x^2 +x^6 = (y^3-x^4)(y-x^2) has 2 branches, one of which is smooth; y^4 –y^2x^3 – y^2x^5 = (y^2-x^3)(y^2-x^5) has two singular branches; (y^4-x^5) is locally irreducible, but all have the same lowest order term y^4. Thus a lowest order term of form y^n gives almost no information about the number or smoothness of the branches if $n \geq 2$, except that the number of branches is $\leq n$.

For monomial singularities, the two lowest terms do determine the local branches

If we have an analytic local equation in the monomial form y^n – x^m however, then we can tell exactly how many local analytic branches there are using both terms, and can write down explicit local parametrizations for each branch. Of course this is because we can actually factor such a simple expression into irreducible factors in the ring of polynomials, and the factors are also irreducible in the ring of power series. We explore the different cases next.

Monomial singularities with only one local branch

If f(x,y) = y^n – x^m where n,m are relatively prime, then there is only one local analytic branch, and the curve is parametrized by the map t--->(t^n,t^m). E.g. the curve y^3-x^8 = 0 is parametrized by t--->(t^3,t^8) = (x,y). Since t = (t^3)^3/t^8 = x^3/y, there is an inverse to this parametrization on a punctured neighborhood of the singularity defined by (x,y)--->x^3/y. To check the composition in the other direction, note that (x,y)--->t = x^3/y---> (x^9/y^3, x^24/y^8), which if we use the equation y^3=x^8, equals (x^9/x^8, y^9/y^8) = (x,y).

In general, if an+bm = 1, the inverse to $t \text{--->} (t^n,t^m)$ is defined by $(x,y) \text{--->} t = (x^a y^b)$. I.e. then $t \text{--->} (x,y) = (t^n,t^m) \text{--->} t^{(an)} t^{(bm)} = t^{(an+bm)} = t$. In the other direction, $(x,y) \text{--->} x^a y^b \text{--->} (x^{(an)} y^{(bn)}, x^{(am)} y^{(bm)})$, and using $y^n = x^m$ gives $(x^{(an)} x^{(bm)}, y^{(an)} y^{(bm)}) = (x^{(an+bm)}, y^{(an+bm)}) = (x,y)$.

Since a punctured neighborhood of the singularity is isomorphic to a punctured disc, there is only one local analytic branch in this case. A small punctured neoghborhood thus is isomorphic to a punctured disc, and the Riemann surface is formed by capping off this punctured disc with a single smooth disc. The Milnor number is $\mu = (n-1)(m-1)$, so a local smoothing of the curve is obtained by instead plumbing into the hole left by the singular point, a compact surface of genus $\mu/2$ from which a single disc has been removed.

A monomial singularity $\{y^n - x^m = 0\}$ has k = gcd(n,m) local analytic branches

If the equation is $y^n - x^m$, where $n = ak \leq m = bk$ with $1 \leq a \leq b$, and a,b relatively prime, then there are exactly k = gcd(n,m) local analytic branches. To see this just factor as in the "ordinary" case where the exponents are equal, i.e. $y^n-x^m = (y^a)^k – (x^b)^k = \prod(y^a - \mu j.x^b)$, as μj runs over all the kth roots of 1. Each factor $\{y^a - \mu j.x^b) = 0\}$ is of the type just shown in the previous paragraph to define a locally irreducible curve, so this factorization into k distinct factors shows the original curve has k distinct local analytic branches, all "tangent to" the line y = 0. These branches are all smooth if and only if a = 1, i.e. if and only if n divides m. Otherwise all branches are singular.

The Milnor number is easily calculated as $\mu = \dim k[[x,y]]/(x^{(n-1)}, y^{(m-1)}) = (n-1)(m-1)$, and the number of branches is k = gcd(n,m). Thus the Riemann surface is formed by separating the k branches and capping off the resulting k holes by k discs. A local smoothing is formed by instead plumbing in a handlebody made from a connected compact surface of genus $(1/2)(\mu+1-r) = (1/2)([n-1][m-1]+1-k)$, by removing k discs. Thus if the curve is globally irreducible, this singularity reduces the genus of the Riemann surface, compared to a nearby smoothing, by $(1/2)([n-1][m-1]+k-1)$.

Tomorrow we discuss a class of global equations, the cyclic covers of P^1, for which all singularities are monomial, and we can compute the local equations at all singularities, and hence the genus from the global poloynomial equation.

The quadratic transform of P^2

Just for fun, we look at a global example of a Riemann surface for a plane quartic with three nodes. This is an example of the famous "quadratic transform" of the plane. Consider the following "rational map" $P^2 \text{--->} P^2$, i.e. the map defined by the rational functions $[x : y : z] \text{--->} [1/x : 1/y : 1/z] = [yz : xz : xy]$. The first version shows that the map is its own inverse where defined. Notice this map, even in its second representation, is undefined where any two coordinates are zero. Moreover on the line where any one coordinate is zero, the map sends the rest of that line to one point. The lines x = 0, y = 0, z = 0 form what is called the "coordinate triangle" in P^2, and this map "blows down" the sides of this triangle, while simultaneously "blowing up" the vertices. I.e. the 3 sides are mapped to the three vertices, and the three vertices are mapped to the three sides.

Points that approach a given vertex along different directions have images that approach different points on the image line for that vertex. This makes the operation good for resolving singularities, since transverse branches of a singularity occurring at a vertex are separated into different image points by the map. In fact although not obvious to me, repeated blowings up eventually separate even tangent branches. I.e. I guess repeated blowings up correspond to taking higher derivatives, and at some level any two distinct branches have distinct derivatives or "jets". We are interested also in the opposite result, namely a curve meeting a coordinate line transversely at two different points will have those two points mapped to the same image point, creating a node on the image curve. A smooth curve meeting one these lines tangentially at a point p, will have the smooth branch of the curve at that point p transformed into a singular branch.

Now look at the smooth conic x^2 + y^2 + z^2 = 0, chosen so as not to pass through any vertex of the coordinate triangle. Its image under the quadratic transform, which equals its pullback by the self inverse map, is the quartic {y^2 z^2 + x^2 z^2 + x^2 y^2 = 0}. This quartic is irreducible since it is the image of an irreducible conic, and one can check the quartic has nodes at each of the three coordinate vertices. Since a smooth quartic has genus three, we expect the Riemann surface of this irreducible 3 nodal quartic to have genus zero. Indeed that is borne out by the explicit parametrization of it by the smooth conic above.

Inversely, the quadratic transform, which is defined except at the nodes of the quartic, maps the quartic birationally to the smooth conic, thus desingularizing it. In general a quadratic transform improves singularities occurring at the coordinate vertices and creates new singularities from points meeting the sides of the coordinate triangle. If those intersections with the sides are transverse however, these new singularities are ordinary ones. By changing coordinates and repeating this process, it can be shown that an arbitrary reduced plane curve can be transformed in a finite number of steps into one with only ordinary singularities. In particular at that point the genus can be calculated. I learned this beautiful classical theory from the book of Walker, Algebraic plane curves, and one can also learn it in modern language from the free book, Algebraic curves, by Fulton. The modern approach in Fulton has some advantages especially in the later discussion of the Riemann Roch theorem, but the chapter on resolving plane singularities by quadratic transforms in Walker is very nice and explicit. It is also reproduced in my class notes from 1991 that I handed out.