8320 Spring 2010 Day 18: Cyclic covers of the projective line

Today we will study a class of Riemann surfaces that have affine equations similar to those of hyperelliptic curves, i.e. $\{y^n = h(x)\}$. These are nice because all their finite singularities are monomial, hence highly calculable. These occur at multiple roots of h. Moreover even at infinity, the singularities can be represented as monomial ones by a trick.

All finite singularities of cyclic covers are monomial

First we note that by introducing units, or by removing them, we can recognize some singularities as monomial that did not look that way originally. A monomial singularity is one with local analytic equation of form $y^n = x^m$. All we need for this is to have the variables separated as in a hyperelliptic equation. I.e. the curve $y^2 = (x^2)(x-1)^3$ has monomial singularities in the finite affine (x,y) plane at the multiple roots of the right hand side, namely at x= 0, and x=1. To see this, look at x=0 first. Here the RHS can be expressed as $x^2 (x^3 -3x^2 + 3x -1)$, and since the second factor is a unit, if take an analytic square root of that unit, and multiply it by x to get v, we have the local analytic equation $y^2 = v^2$, which is monomial with two transverse branches. At x=1, the other factor x^2 is a unit, since $x^2 = (x-1+1)^2 = 1 +2(x-1) + (x-1)^2$, or more obviously since x^2 does not vanish at x=1. Taking a local analytic cube root of x^2 near x=1, and multiplying it by (x-1) to get u, changes our local equation into $y^2 = u^3$, a monomial singularity with one singular branch. In general then $y^n = (x-a1)^{m1} (x-a2)^{m2} \ldots (x-ak)^{mk}$, is monomial near aj, with local analytic equation $y^n = v^{mj}$, which we know how to analyze from yesterday. For example, $\{y^3 = x^2 (x-1)^3 (x+1)^4\}$, has one branch at x=0, 3 branches at x=1, and one branch at x= -1.

Points "at infinity" of cyclic covers have monomial realizations

If we use our usual method of projective completion, then the singularity at infinity of a cyclic cover $\{y^n = h(x)\}$ looks monomial at least when h has a simple equation like $y^3 = h(x) = x^5 – 1$. Then the projective completion is $y^3 z^2 = x^5 – z^5$. The unique point on z=0 is [0: 1: 0], and viewed in the affine open set y=1 this looks like $z^2 + z^5 = x^5$, which is the monomial singularity $z^2 = x^3$ multiplied on the left by the unit $(1 + z^3)$. But even a slightly more complicated h(x) like $y^3 = h(x) = x^2 + x^5 - 1$, already has a singularity at infinity that is harder to analyze in the projective plane. I.e. then the projective equation is $y^3 z^2 = x^2 z^3 + x^5 – z^5$, which in y=1 becomes $z^2 + z^5 = x^5 + x^2 z^3$. Since the variables are not separated here, it does not look monomial, and it is harder to analyze the branches.

Rick uses a trick to represent the Riemann surface of this affine curve in a different way. Instead of forming the projective completion, he forms instead another affine cyclic cover $w^n = g(z)$, which is isomorphic to this one, but with the transform z= 1/x, so that the points at x= infinity of the original curve, become the points at z=0 on the new curve. He then glues them together. Thus the Riemann surface is the union of two affine cyclic covers of the same form and the singularities can all be analyzed as monomial ones as above. One must however take some pains to be sure the two affine cyclic covers really are isomorphic, or they will not glue, and this I did not do correctly in class on Friday. Let me try to explain that in these notes.

Suppose we want to compactify the curve {y^3 = x^4 – 1}, which is a triple cover of the complex line, branched at the 4 roots of unity. If we use the coordinate z = 1/x for these 4 roots of unity instead, then the equation w^3 = (1-z^4) is also a triple cover of the line branched over the reciprocals of the 4 roots of x^4-1= 0. But that is not enough to glue them, i.e. the two covers are not isomorphic when x = 1/z! Why not? This is because the first equation {y^3 = x^4 – 1}is also branched at infinity, so if we seek to identify x= infinity with z = 0, we need to use an equation in z and w that is branched at z=0 in the same way that {y^3 = x^4 – 1} is branched at x=infinity.

In terms of topology, i.e. fundamental groups and monodromy, there is a loop around the point at x= infinity represented by any large circle in the x line. As an element of the fundamental group of the line minus the 4th roots of unity, that large circle is homotopic to the product of the 4 small loops, one around each root of unity. But in a cyclic triple cover like this one, each time we go around a root of unity, i.e. a branch point, we cycle up one level on the three levels of our triple cover. Thus after going around all 4 roots of unity we have not returned to our original position on the triple cover, but we are one level off. Thus going around infinity changes levels also.

Hence the affine curve w^3 = (1-z^4) is wrong because it is unbranched at z=0, since it has 3 distinct points over z=0, namely the three roots of w^3 = 1. To get an affine curve that is not only branched at the 4^{th} roots of 1, but also at z=0, in the same way that {y^3 = x^4 – 1} was branched at x = infinity, we need to introduce some branching at z=0 into our equation w^3 = g(z). The correct equation is w^3 = z^2(1-z^4). I.e. geometrically we have changed the orientation of a loop around x = infinity by replacing it with a loop around z=0, so we need to go twice around z=0 instead of going 3 + 1 time around x = infinity.

Algebraically, the correct equation is even easier to see, as pointed out by Jennifer. We want to substitute correctly x = 1/z, so y^3 = (x^4 – 1) becomes y^3 = (1/z^4 – 1), which is true (when z≠0) if and only if: z^4y^3 = (1 – z^4). Now we want to take a cube root, but we also want it to be a rational function of x and y so the two curves will be algebraically isomorphic, so we cannot take a cube root of z^4y^3 = y^3/x^4 since y x^(-4/3) is not analytic on any neighborhood of x= 0. So we must multiply through by some power of z that makes the expression on the left a perfect cube, say z^2, getting z^6y^3 =z^2 (1 – z^4). Now setting w = yz^2 = y/x^2, gives w^3 = z^2(1-z^4) as our equation. I.e. this is a triple cover of the punctured z line that is branched over the 4 roots of unity z = 1/x, exactly the way {y^3 = x^4 – 1} was branched at the corresponding points x, but now this new cover is also branched at z=0 in the same way that {y^3 = x^4 – 1} was branched at x = infinity. Thus they paste together. To actually paste them we just map (x,y)--->(z,w) = (1/x, y/x^2), with inverse map (z,w)--->(x,y) = (1/z, w/z^2). When xz ≠ 0 these substitutions were chosen to make the curve {y^3 = (x^4-1)} transform into the curve {w^3 = z^2(1-z^4)} and vice versa. (Please check it.)

Now interestingly this solution is not unique. I.e. we could have overdone it and multiplied the equation z^4y^3 = (1 – z^4) through by z^5 getting z^9y^3 = z^5(1 – z^4), hence w^3 = z^5(1-z^4) with w = yz^3. This is actually branched over z=0 the same way as the previous curve. The two curves we constructed in (z,w) are not isomorphic at z=0, but their Riemann surfaces are, because those are determined by the covering space map of the punctured surface. E.g. at z=0, the monomial singularity w^3 = z^2(1-z^4), has one branch, as does the monomial singularity of w^3 = z^5(1-z^4), so both Riemann surfaces are constructed by adding one smooth point to the same smooth punctured surface.

Some genus calculations

The point is that we should be able to compute the genus of the Riemann surface of any curve of form {y^n = h(x)}. i guess we should worry about irreducibility, but i guess these are irreducible if and only if, hmmm, at least if the multiplicity of some root of h is relatively prime to the degree n of the LHS. I haven't really thought about this, but it seems clear that if n is prime for example, and some root of h has multiplicity not divisible by n, then the cover is connected so the polynomial is irreducible.

So let's do one, first the one I got wrong in class: {y^3 = x^4 – 1}. This one is easy by projectivizing. First off we have f(x,y) = y^3 – x^4 + 1, so ∂f/∂y = 2y, which equals zero only at y=0. Thus we have a singular point at (x,y) if and only if y=0 and ∂f/∂x = ∂(x^4-1)/∂x = 0, which never happens, since x^4-1 has no multiple roots. I.e., y^n = h(x) has a singularity at (x,y) if and only if y = 0 and x is a multiple root of h(x). So the map sending (x,y)--->x is 3 to 1, and branched in the finite part of the line only at the 4 roots of x^4-1=0, with ramification index 2 at each such root. Now we look at x = infinity. I.e. we projectivize the curve to get zy^3 = x^4 – z^4, and set z=0, hence x=0 so y=1. Thus there is only on point at z=0, so in affine coordinates there we set y = 1, and get affine equation z = x^4 –z^4. Now at (x,z) = (0,0), this has a non zero linear term z. So this is a non singular point. What we care about is that there is only one analytic branch. So there is only one point of the Riemann surface over x= infinity, hence it must have ramification index 2. Thus we have all told 5 ramification points, each with ramification index 2, so if we let R=10, then the genus g(X) of the Riemann surface X satisfies 2-2g(X) = 3(2 – 2g(P^1)) – 10. I.e. 2 – 2g(X) = -4, so 6 = 2g(X) and g(X) = 3. On the other hand, we did not find any singular points either in the finite part of the plane or at infinity, so this is a non singular projective quartic, which should have genus g = (1/2)(3)(2) = 3.

Now lets use Rick's trick of capping it off at infinity by another triple cover instead of taking the projective closure. I.e. we are just trying to understand the compactification of the affine curve and we can compute that compactification in any way that is convenient. So we substitute x = 1/z, and get y^3 = (1/z^3 – 1). Now in the finite part of the plane where z≠0, this holds if and only if the equation holds after multiplying through by z^4, i.e. y^3z^4 = (1-z^4). Now we want to take a cube root of the LHS, and this requires the LHS be a perfect cube, so multiply further by z^2 on both sides, getting y^3z^6 = z^2(1-z^4), and set w = yz^2, so finally w^3 = z^2(1-z^4). As we checked above, this curve is isomorphic to the previous one on the open sets where xz≠0. And on this curve, the point z=0 corresponds to x= infinity on the previous curve. So to understand what was happening at x = infinity on that curve, we look at z=0 on this one. Well, z=0 implies w=0 so the only point is (z,w) = (0,0), at which we have a monomial singularity of type (3,2). That has only one branch, so the Riemann surface has only one point over z=0, i.e. only one point over x = infinity. So again this point is a ramification point of index 2, and we get, thankfully, the same calculation of the genus as by the projective method, namely 5 ramification points, each with index 2, so 2-2g = 6 – 10, and

$g = 3$. Notice we got two different curves by our two methods, one non singular, and one singular. But we were interested in the desingularization of those curves, i.e. in the Riemann surfaces, which were the same.

[What if we ask about the two "smoothings"? Well then we have to say what we are talking about when we say "smoothing". A smoothing only makes sense when a curve belongs to a specific family of curves. In the first case above our plane curve belongs to the family of projective plane curves of degree 4, and it is already smooth, i.e. a general curve in that family is smooth of genus 3. In the second case I guess we could make a family of pairs of glued triple covers, but it seems we would have to use triple covers with 6 ramification points each of index 2, hence the general smooth one would seem to have genus 4. Does this make sense?]

If we go back to day 3, we began to lose our intuition on the examples y^3 = x^3-1 and y^4 = x^4-1. These should be easy now by either of our methods. I.e. both are non singular in the affine plane, and projectivizing the first gives y^3 = x^3-z^3, with three points on the line z=0 at infinity. Thus x = infinity is not a branch point of the triple cover (x,y)--->x, and we have three branch points each of index 2, so R = 6, and we have 2-2g = 6-6 = 0 so g = 1. As for y^4 = x^4-1, we get 4 points at infinity all unbranched, so we have R = 8 and 2-2g = 6 -8 = 02, so g= 3. Lets try Rick's trick on these. We get from y^3 = x^3 – 1, that x = 1/z implies z^3y^3 = 1 – z^3, so w^3 = 1-z^3, which has three unbranched points on the line z=0. Again we have R = 6, g = 1. And for y^4 = x^4-1, x= 1/z gives z^4y^4 = 1-x^4, so w^4 = 1-x^4, again 4 points at infinity, so R = 8, g = 4. Good, all is right with our little world.

One more: y^6 = x^3(x-1)^4(x+1)^2(x-2)^5. Here we have four monomial singularities in the (x,y) plane, at x=0,1,-1,2. At x=0 we have 3 branches hence ramification index 6-3 = 3 on this fiber. At x=1 we have 2 branches, hence ramification index 6-2 = 4 on this fiber. At x= -1 we have 2 branches, hence ramification index 6-2 = 4 on this fiber. At x = 2, we have 1 branch, hence ramification index 6-1 = 5. Thus the total ramification in the (x,y) plane is 3+4+4+5 = 16. At infinity we substitute x =1/z, getting z^14 y^6 = (1-z)^4 (1+z)^2 (1-2z)^5. This becomes z^18 y^6 = w^6 = z^4(1-z)^4 (1+z)^2 (1-2z)^5, so at z=0 we have a monomial singularity with 2 branches, hence index 6-2 = 4. Thus in all R = 16+4 = 20, so we have 2-2g = 6 – 20, so g=8. Does this check?

Just for the heck of it let's try an easier one that should behave the same at infinity, by the projective method. I.e. try y^6 = (x^14 – 1), but restrict ourselves to computing the behavior at infinity. Then we have at infinity, z^8 = x^14 – z^14, which has 2 branches at infinity. If we assume our original curve y^6 = h(x) where h has degree 14, has the same behavior at infinity, then this calculation agrees with the previous one. I.e. I conjecture that every cyclic cover of form y^n = h(x) where h has degree k, behaves at infinity topologically the same as the easy one y^n = x^k – 1. Well I guess this is proved by the cyclic cover trick, since that calculation shows the number of branches at infinity is determined just by n and the degree of h. In fact it seems to be gcd(n,deg(h)).