8320 Spring 2010, day one Introduction to Riemann Surfaces

We will describe how Riemann used topology and complex analysis to study algebraic curves over the complex numbers. [The main tools and results have analogs in arithmetic, which I hope are more easily understood after seeing the original versions.] The idea is that an algebraic curve C, say in the plane, is the image by a holomorphic map, of an abstract complex manifold, the Riemann surface X of the curve, where X has an intrinsic complex structure independent of its representation in the plane.
We will construct two fundamental functors of an algebraic curve, the Riemann surface X, and the Jacobian variety J(X), and natural transformations X^(d)--->J(X), the Abel maps, from the "symmetric powers" X^(d) of X, to J(X).

The Riemann surface X
The first construction is the Riemann surface of a plane curve:
{irreducible plane curves C: f(x,y)=0} ---> {compact Riemann surfaces X}

The first step is to compactify the affine curve C: f(x,y) =0 in A^2, the affine complex plane, by taking its closure in the complex projective plane P^2. Then one separates intersection points of C to obtain a smooth compact surface X. X inherits a complex structure from the coordinate functions of the plane. If f is an irreducible polynomial, X will be connected. Then X will have a topological genus g, and a complex structure, and will be equipped with a holomorphic map *f*:X--->C of degree one, i.e. *f* will be an isomorphism except over points where the curve C is not smooth, e.g. where C crosses itself or has a pinch.

This analytic version X of the curve C retains algebraic information about C, e.g. the field M(X) of meromorphic functions on X is isomorphic to the field Rat(C) of rational functions on C, the quotient field k[x,y]/(f), where k = complex number field. It turns out that two curves have isomorphic Riemann surfaces if and only if their fields of rational functions are isomorphic, if and only if the curves are equivalent under maps defined by mutually inverse pairs of rational functions. Since the map X--->C is determined by the functions (x,y) on X, which generate the field Rat(C), classifying algebraic curves up to "birational equivalence" becomes the question of classifying these function fields, and classifying pairs of generators for each field, but Riemann's approach to this algebraic problem will be topological/analytic. We already can deduce that two curves cannot be birationally equivalent unless their Riemann surfaces have the same genus. This solves the problem that interested the Bernoulli's as to why most integrals of form dx/sqrt(cubic in x) cannot be "rationalized" by rational substitutions. I.e. only curves of genus zero can be so rationalized and y^2 = (cubic in x) usually has positive genus.

The symmetric powers X^(d)
To recover C, we seek to encode the map *f*:X--->C, i.e. *f*:X--->P^2, by intrinsic geometric data on X. If the polynomial f defining C has degree d, then each line L in the plane P^2 meets C in d points, counted properly. Thus we get an unordered d tuple of points L.C, possibly with repetitions, on C, hence when pulled back via *f*, we get such a d tuple called a positive "divisor" D = *f*^(-1)(L) of degree d on X. (D = n1p1+...nk pk,

where nj are positive integers, n1+...nk = d.) Since lines L in the plane move in a linear space dual to the plane, and (if d ≥ 2) each line is spanned by the points where it meets C, we get an injection P^2*--->{unordered d tuples of points of X}, taking L to *f*^(-1)(L).

If X^d is the d - fold Cartesian product of X, and Sym(d) is the symmetric group of permutations of d objects, and we define X^(d) = X^d/Sym(d) = the "symmetric product" of X, d times, then the symmetric product X^(d) parametrizes unordered d tuples, and inherits a complex structure as well. Thus the map *f*:X--->C yields a holomorphic injection P^2*--->∏of the projective plane into X^(d). I.e. the map *f* determines a complex subvariety of X^(d) isomorphic to a linear space ∏ ≈ P^2*. Now conversely, this "linear system" ∏ of divisors of degree d on X determines the map *f* back again as follows:

Define *f*:X--->∏* = P^2** =P^2, by setting *f*(p) = the line in ∏ consisting of those divisors D that contain p. Then this determines the point *f*(p) on C in P^2, because a point in the plane is determined by the lines through that point. [draw picture]

Thus the problem becomes one of determining when the product X^(d) contains a holomorphic copy of P^2, or copies of P^n for models of X in other projective spaces.

The Jacobian variety J(X) and the Abel map X^(d)--->J(X)
For this problem, Riemann introduced a second functor the "Jacobian" variety J(X) = k^g/lattice, where k^g complex g -dimensional space. J(X) is a compact g dimensional complex group, and there is a natural holomorphic map Abel:X^(d)--->J(X), defined by integrating a basis of the holomorphic differential forms on X over paths in X. Abel collapses each linear system ∏ ≈ P^n* to a point by the maximum principle, since the coordinate functions of k^g have a local maximum on the compact simply connected variety ∏. Conversely, each fiber of the Abel map is a linear system in X^(d).

Existence of linear systems ∏ on X: the Riemann - Roch theorem.
By dimension theory of holomorphic maps, every fiber of the abel map X^(d)--->J(X) has dimension ≥ d-g. Hence every positive divisor D of degree d on X is contained in a maximal linear system |D| , where dim|D| ≥ d-g. This is called Riemann's inequality, or the "weak" Riemann Roch theorem.

The Roch part analyzes the relation between D and the divisor of a differential form to compute dim|D| more precisely. Note if D is the divisor cut by one line in the plane of C, and E is cut by another line, then E belongs to |D|, and the difference E-D is the divisor of the meromorphic function defined by the quotient of the linear equations for the two lines. If D is a not necessarily positive divisor, we define |D| to consist of those positive divisors E such that E-D is the divisor of a meromorphic function on X. If there are no such positive divisors, |D| is empty and has "dimension" equal to -1. Then if K is the divisor of zeroes of a holomorphic differential form on X, the full Riemann Roch theorem says: dim|D| = d-g +1+dim|K-D|, where the right side = d-g when d > deg(K).