8320 sp2010 day 4. Projective line P^1 and projective plane P^2

P^1: We want to compactify the complex line k and complex plane k^2 in a natural way, obtaining compact complex manifolds P^1, P^2 of complex dimension one and two respectively. The complex line is compactified by adding just one point "at infinity", to form P^1 = k + {infinity}. This is done by taking another copy of the complex line and gluing the two copies together on the complement of their origins by the reciprocal map. I.e. let s be an affine coordinate for one copy of the complex line and let t be an affine coordinate for another copy of the complex line. Then identify the open sets {s≠0} and {t≠0} by mapping s to t = 1/s. This is a holomorphic isomorphism, and if we form the identification space from the disjoint union of these two copies of the complex line, joined along those two isomorphic open sets, we get a compact Hausdorff space called P^1, the complex projective line. Moreover since the map s--->1/s is an invertible holomorphic map, this space P^1 has an open cover by compatible holomorphic coordinate charts making it into a complex one - manifold, or compact Riemann surface. To see P^1 is compact it suffices to note it is the union of the two compact subsets {|s|≤1} and {|t|≤1|}. To see it is Hausdorff, since the s line k is Hausdorff, it suffices to note that the additional point t=0 in P^1, can be separated from any point s0 by the open sets, |s| < 2|s0|, and |s| > 2|s0|, i.e. |t| < 1/|2s0|.

There is another construction of P^1 by starting from the complex affine plane k^2, and considering P^1 to be the set of lines through the origin. I.e. take the punctured affine plane k^2 –(0,0), and define an action of k* = non zero complex numbers on it, by saying that c in k* sends (a,b) in k^2 – (0,0) to (ca,cb). Then the orbits of this action are exactly the punctured lines through the origin of k^2. The quotient space [k^2-(0,0)]/k* is also P^1. I.e. if we take coordinates (z,w) on k^2, and use [z:w] to represent the equivalence class of (z,w) in P^1, then we can cover this manifestation of P^1 by the two open sets {z≠0} and {w≠0}. The first open set {z≠0} is isomorphic to the t line by the map s--->[1:t] = [z:w] from k to P^1, with inverse map [z:w]--->(w/z) = t. The second open set {w≠0} is isomorphic to the s line by the map s--->[s:1] = [z:w], with inverse map [z:w]---> (z/w) = s. In the overlap zw≠0, a point [z:w] of P^1 has s and t coordinates related by s = (z/w) = 1/(w/z) = 1/t. Hence this is isomorphic to the identification space version of P^1 constructed above from identifying the s line and t line by reciprocation.

This gives another way to see P^1 is compact since the 3 sphere in k^2 ≈ R^4, real 4 space, surjects onto it. I.e. every complex line in k^2 is a real plane in R^4 and meets the 3 sphere in a circle. Thus the map S^3--->P^1 is a circle bundle over the sphere P^1, called the "Hopf map". This maps represents a non zero generator of the higher homotopy group π3(S^2), a rather fascinating object since it seems strange for a 2 sphere to admit a homotopically non trivial map from a 3 sphere. There are no non trivial maps from higher spheres to S^1 for instance because the universal cover of S^1 is contractible. The classification of all homotopically non - trivial maps from higher dimensional spheres to lower dimensional spheres is still an open problem.

Since P^1 is homeomorphic to a sphere, it has genus zero. Moreover, this is the only compact topological surface that can be constructed from the complex line k by adding a finite number of points, hence the Riemann surface of any affine curve homeomorphic to k also has genus zero. E.g. any affine line {ax+by = 0}, a,b not both zero, must have a Riemann surface of genus zero. Of course the proof of this must await the definition of the Riemann surface associated to an affine plane curve.

We can also give the construction of P^1 in a coordinate - free way, by starting from any 2 dimensional complex vector space V and considering the set P(V) of all lines through the origin. Note that if (a,b) and (c,d) are non parallel vectors in k^2, then the map taking (z,w) --->(az+bw, cz+dw) = (u,v) is an isomorphism of k^2, and takes the set{az+bw≠0} to the set {u≠0}, and takes the set {cz+dw≠0} to the set {v≠0}. Moreover in local holomorphic coordinates, as a map say from an open subset of {z≠0} to {v≠0}, in the local coordinates t = w/z, and s = u/v, this map has form s = [a+bt]/[c+dt] where c+dt≠0, hence defines a holomorphic bijection thus an isomorphism of P^1. It follows that we could recoordinatize P^1 as the union of any two affine open sets u = az+bw ≠ 0, and v = cz+dw ≠ 0. In particular any choice of vector basis for V yields the same holomorphic structure on P(V) ≈ P^1.

P^2: Now we want to compactify the complex plane k^2 to obtain a compact complex two-manifold P^2. First of all we may define a holomorphic function on a open subset of k^2 to be a continuous function which is holomorphic in both variables separately. [In fact continuity is guaranteed by the separate holomorphicity.] Now we will compactify k^2 by adding in a copy of P^1 to form P^2. Thus if we think of k^0 as one point, then P^2 = k^2 + k^1 + k^0, as disjoint union of sets. The coordinate definition is to start from k^3 and consider all lines through the origin. I.e. on k^3 – (0,0,0) define the k* action of scaling as before, with a non zero scalar c sending (z0,z1,z2) to (cz0,cz1,cz2). The quotient space by this action [k^3 – (0,0,0)]/k* = P^2. It has an open cover by three affine open sets {z0≠0}, {z1≠0}, {z2≠0}, with local holomorphic coordinates on {z0≠0} given by (z1/z0, z2/z0), which defines a homeomorphism from P^2 with its quotient topology, to k^2. Composition of these local coordinates gives holomorphic rational functions in the affine coordinates with non zero denominators, hence the local coordinates are holomorphically compatible and we have a complex 2 manifold, analogous to our definition of a complex Riemann surface or complex one manifold.

Since complex 3 space k^3 ≈ R^6 = real 6 space, and every complex line through the origin of complex 3 space, is a real plane through the origin of R^6, which meets the 5-sphere in a circle, the 5-sphere maps surjectively onto the quotient space P^2. Since the sphere is compact, so is P^2.

Homogenizing an affine equation: Now we can take the first step toward defining the Riemann surface of a plane curve, by compactifying the curve. I.e. we will take the closure in P^2 of an affine plane curve in k^2. Suppose {f(X,Y) = 0} is a plane curve in the (X,Y) plane k^2. Think of X,Y as affine coordinates X = Z1/Z0, Y = Z2/Z0 in P^2, and write the equation for the curve as f(Z1/Z0, Z2/Z0) = 0. Then multiply by just a high enough power of Z0 to clear denominators; e.g. if f has degree d, we obtain Z0^d f(Z1/Z0, Z2/Z0) = F(Z0,Z1,Z2). Then this polynomial F will be homogeneous of degree d in the three variables Z0,Z1,Z2. Then the zero locus of F in P^2, which makes sense because a homogeneous equation vanishes on $[a:b:c]$ if and only if it vanishes on $[ta:tb:tc]$, is just the closure in P^2 of the zero locus of the affine equation f. Moreover F is irreducible if and only if f is.

Example: Start from $f(X,Y) = Y^2 - X^4 - 1$ of degree 4. This is essentially the same topologically as problem #4, day 3, hence is a non singular affine curve whose Riemann surface is obtained by adding two points to this affine curve, and has genus one. But is this visible from the compact projective version? Homogenizing gives $F(Z0,Z1,Z2) = Z0^4[Z2^2/Z0^2 - Z1^4/Z0^4 - 1] = Z0^2Z2^2 - Z1^4 - Z0^4$, which is homogeneous of degree 4. Now the original affine curve was the part of this projective curve lying in the open set $\{Z0\neq 0\}$. Thus the added points lie on the line $Z0=0$. Setting $Z0=0$ in this new equation implies $Z1 = 0$, and hence $Z2 \neq 0$, since in projective space some coordinate must be non zero. Since coordinates may be scaled in P^2, there is only one added point, $[0:0:1]$. To see whether this point is non singular we can use a different set of affine coordinates. I.e. since $Z2 \neq 0$ at this point, we can look in the affine open set $\{Z2 \neq 0\}$, where affine coordinates are given by $u = Z0/Z2$, $v = Z1/Z2$. Thus in these coordinates, the equation becomes $(1/Z2^4)[Z0^2Z2^2 - Z1^4 - Z0^4] = 0$, or $[(Z0^2/Z2^2) - (Z1^4/Z2^4) - (Z0^4/Z2^4)] = 0$, or $u^2 - v^4 - u^4 = 0$. Since the point $[Z0:Z1:Z2] = [0:0:1]$ has affine coordinates $(u,v) = (0,0)$, this is a singular point.

To compute the number of local analytic branches we can proceed as follows. This equation can be written as $u^2 - u^4 - v^4 = u^2(1-u^2) - v^2 = 0$. Since near $(0,0)$, $1-u^2$ is an analytic unit, we can take a square root e of it and take as new local analytic coordinate $t = u.e$. In this coordinate, the equation becomes $t^2 - v^4 = 0$, which factors locally as $(t-v^2)(t+v^2) = 0$. Hence there are two local analytic branches $t = v^2$ and $t = -v^2$, both tangent to each other, but distinct. Hence the Riemann surface has two distinct points in place of this singular point. This is what we "saw" from our explicit pictures earlier for similar examples, when we concluded the genus was one. Thus the projectivization only helped us find the number of missing points, but we will soon have a way to compute the genus of this example as well, using Milnor numbers of isolated singular points.

Remark: Sometimes we are careless with homogeneous coordinates and their affine counterparts, using the same letter for both. This can be confusing but is often quicker for some calculations. For instance in the previous example, starting from affine equation $y^2 - x^4 - 1 = 0$, we could have "homogenized" just by boosting all monomials up to degree 4, adding one new letter, getting $y^2 z^2 - x^4 - z^4 = 0$. Then $z=0$ implies $x = 0$, so $y = 1$, and we have the point $[x:y:z] = [0:1:0]$. Here $y=1$, so we set $y=1$ and take affine coordinates x,z, getting equation $z^2 - z^4 - x^4 = 0$. This is essentially equivalent to the equation above, or at least it gives the same local analysis of the point $(x,z) = (0,0)$. Of course x means something different in all three equations here. First it represents $Z1/Z0$, then it represents $Z1$, and finally it represents $Z1/Z2$. So be careful with this.

The homogeneous implicit function theorem

There is another way to check singularity of a point on a projective curve directly from the homogeneous equation, without using affine coordinates. I.e. a point on a projective curve with homogeneous equation $F(Z0,Z1,Z2) = 0$ is singular if and only if all partials $\partial F/\partial Zj = 0$ at the point. In the previous example we could have thus used the equation $F = Z0^2Z2^2 - Z1^4 - Z0^4$, with partials $\partial F/\partial Z0 = 2Z0.Z2^2 - 4Z0^3$, $\partial F/\partial Z1 = -4Z1^3$, and $\partial F/\partial Z2 = 2Z0^2Z2$. All these partials vanish at [0:0:1], so this is a singular point of the curve. In fact in the homogeneous case, it follows from the vanishing of the partials that the homogeneous polynopmial itself vanishes at the point, so we do not even have to check that the point is on the curve. I.e., if all $\partial F/\partial Zj = 0$ at a point then F also vanishes at that point. This follows from "Euler's theorem", the easy result that $d.F = Z0.\partial F/\partial Z0 + Z1.\partial F/\partial Z1 + Z2.\partial F/\partial Z2$. This can be checked on monomials $F = Zj^d$ since both sides are linear in F.

To see that the homogeneous criterion for singularity is equivalent to the affine one is not hard either using the chain rule and the basic fact that the gradient of a function is perpendicular to the level sets of the function. E.g. in the open set $Z0 \neq 0$, we have affine coordinates X,Y and a parametrization defined by sending (X,Y)--->[1:X:Y] = {Z0,Z1,Z2]. If F(Z0,Z1,Z2) is the homogeneous equation of our curve, the affine equation of this piece of it is f(X,Y) = F(1,X,Y). Then by the chain rule, if all partials $\partial F/\partial Zj$ vanish at a point, then all partials $\partial f/\partial X$, $\partial f/\partial Y$ vanish at the affine coordinates (X,Y) of that point. Conversely if both affine partials vanish at a point [1:x0:y0], then by the chain rule at least $\partial F/\partial Z1$, and $\partial F/\partial Z2$ vanish there. If $\partial F/\partial Z0$ did not vanish at this point, then $\partial F/\partial Z0$ has form [a,0,0] where $a \neq 0$. But $gradF = (\partial F/\partial Z0,\partial F/\partial Z1,\partial F/\partial Z2)$ is perpendicular to the level set of F which contains the line (t,x0,y0), for all t. Thus gradF should be perpendicular to the tangent vector (1,0,0) of this line, a contradiction, since (1,0,0) is not perpendicular to (a,0,0) unless a = 0.

Projective n space is defined exactly like P^2, as the quotient space of k^(n+1)-(0,...,0) by the k* action of scaling the coordinates. It is covered by n+1 holomorphically compatible affine open subsets each homeomorphic to k^n, and has thereby the structure of a complex manifold of dimension n. We can define curves in P^n for n > 2 also but it requires at least (n-1) equations to do so in P^n.