

8320 Day 6, Wed Jan 20, 2010 Milnor numbers & the genus of a Riemann surface

We are assuming a big theorem, the existence of the Riemann surface of an reduced projective plane curve. I.e. let $F = F_1 \dots F_n$ be a product of distinct irreducible homogeneous polynomials in (x, y, z) . The curve they define in P^2 is called a reduced projective plane curve. It is the union of the irreducible components $F_j = 0$. The Riemann surface of $F=0$ is the disjoint union of the Riemann surfaces of each of the F_j , so it suffices to state the result about an irreducible curve $C: \{F=0\}$. If F is irreducible, there is a compact connected Riemann surface X and a holomorphic map $X \rightarrow C$ which is an isomorphism over the open set of non singular points of C . For each singular point p of C , a small punctured neighborhood of p is homeomorphic to a non empty disjoint union of a finite set of punctured discs. The number of these discs is called the number of local branches at p . There are exactly as many inverse images of p in X as there are local branches at p on C .

Thus constructing X from C is a matter of separating all the local branches at each singular point, and then smoothing the branches out if necessary. I.e. a singularity can arise because several non singular branches intersect at p , or because one singular branch occurs at p , or a combination of these phenomena. The map $X \rightarrow C$ is bijective if and only if at each point there is only one branch, and is an isomorphism if and only if all points of C are non singular. Of course if a curve C is non singular, then it equals its own Riemann surface, so we can say we have constructed the Riemann surface in these cases, using the implicit function theorem to define a coordinate cover. We have not actually constructed any Riemann surfaces from singular curves, but assuming they exist, we have been using the description above to study the genus of the Riemann surfaces of various singular and non singular curves.

I.e. we are trying to understand the Riemann surface associated to an irreducible complex projective plane curve, a compact connected orientable real surface with a holomorphic structure. So first of all we want to compute its topological genus. Without proof, we are going to state some powerful facts about the topology of complex plane curves that is classical, but has become known as the theory of Milnor numbers and Milnor fibers, because Milnor made it precise and generalized it to higher dimensional complex hypersurfaces in a highly recommended pamphlet, Singularities of complex hypersurfaces. Other good references include Algebraic plane curves, by Brieskorn and Knorrer, and Singularities of plane curves, by C.T.C.Wall.

We want to discuss how plane curves vary in families, especially as they acquire isolated singular points. It turns out that non singular plane curves in nice families do not vary at all topologically, but they can vary holomorphically. If a plane curve acquires an isolated singular point however, the topology of the curve will change, and the genus of the associated Riemann surface will go down. We want to compute how much it goes down. Then we can calculate the genus of our Riemann surface from knowing the genus of a non singular curve of the same degree, and then computing how much the genus of our curve is diminished by its singularities. This is determined by an invariant called the Milnor number μ , plus the number r of irreducible local analytic branches of the curve near the singularity. In the other direction, it can be easier to understand the genus of a singular curve than a non singular one, so we can also use a sort of inductive procedure to go back from the genus of a singular curve to the genus of a non singular one, again by knowing how much the genus changes due to the singularities. E.g. a line, a curve of

degree one, is isomorphic to P^1 , which is topologically a sphere, and every curve of degree d can be thought of as obtained by removing the singularities of a set of d general lines. So if we know the genus of a line, and how the genus changes due to a simple singularity like two lines crossing, we can work backwards and learn the genus of any non singular curve of degree d . Here is the answer. If we define the genus of a smooth surface as the unique g such that the Euler characteristic is $2-2g$, then d disjoint lines have genus $1-d$. Then smoothing an intersection of two lines is like forming a connected sum of two disjoint spheres, hence increases the genus by one. Since d lines cross pairwise at $(1/2)d(d-1)$ points, smoothing all self intersections results in a genus for a smooth curve of degree d of $(1-d) + (1/2)d(d-1) = (1/2)(d-1)(d-2)$.

In general the main result is that the genus of the Riemann surface of an irreducible curve is completely determined by the degree and the singularities of the curve as follows. First of all, every non singular plane curve of degree d has the same genus $pa(d)$. Each singularity lowers this genus by a positive amount determined by the local analytic equation at the singularity. Thus our computation of the genus of a plane curve can be done by computing the genus of a non singular curve of degree d , and computing the amount of the decrease due to each singular point of our given curve.

Every singular point p on an irreducible plane curve C has a neighborhood homeomorphic to a “one point union” of discs joined at their centers. The number of these discs is called the number of local analytic branches of C at p . We will see that a non singular complex projective plane curve of degree d has genus $pa(d) = (1/2)(d-1)(d-2)$. For the decrease due to a given singular point, assume C has irreducible affine equation $f(x,y) = 0$, and $(0,0)$ is the given singular point, i.e. $f = \partial f/\partial x = \partial f/\partial y = 0$ at $(0,0)$. The Milnor number of the singularity $\mu = \dim(C[[x,y]]/(\partial f/\partial x, \partial f/\partial y))$, the dimension as (finite dimensional) complex vector space. Then the decrease in the genus due to this singularity is $\partial(p) = (1/2)(\mu + r - 1)$ where r is the number of local analytic branches of the curve at the singularity.

Thus the genus of the Riemann surface of an irreducible complex projective plane curve C of degree d equals $pa(d) = (1/2)(d-1)(d-2)$, if C is non singular, and equals $pa(d) - \partial$, where $\partial =$ the sum of the $\partial(p)$, summed over the finite set of singularities p of C . In particular the genus of the Riemann surface of an irreducible plane curve C can be computed from the degree of C , and the number of local analytic branches and the Milnor number for each singularity of C .

More precisely, a topological model of a non singular curve of degree d can be constructed from any irreducible curve C of degree d as follows. For each singularity p of the given curve C , remove an open neighborhood of the singularity. This neighborhood has a boundary homeomorphic to a disjoint union of r circles, where r is the number of local analytic branches at p . Then glue in place of that neighborhood of p , a compact connected orientable manifold H with boundary homeomorphic to r circles, and with 1st homology group of rank μ . H has the homotopy type of a wedge of μ circles.

For example, if we define an ordinary double point (odp) as a point p where $f = \partial f/\partial x = \partial f/\partial y = 0$, and the Hessian matrix of second partials evaluated at p is invertible, then by the inverse function theorem $(\partial f/\partial x, \partial f/\partial y)$ is a local analytic coordinate system for k^2 at p , where $k =$ the complex number field, so $(\partial f/\partial x, \partial f/\partial y)$ generates the maximal

ideal of the ring of formal power series $k[[x-x(p), y-y(p)]]$ at p , so the Milnor number $\mu(p) = \dim(k[[x-x(p), y-y(p)]] / (\partial f / \partial x, \partial f / \partial y)) = 1$. Then $\partial(p) = (1/2)(\mu + r - 1) = (1/2)(1 + 2 - 1) = 1$. I.e. for any ordinary double point p , $\partial(p) = 1$. Thus the genus of a plane curve C is reduced by one from that of a smooth curve of the same degree, by each odp on C .

For example, we have seen by studying the topology of branched double covers that (the projective closure of) $\{y^2 = x(x^2 - 1)\}$ is non singular of genus one. This confirms our formula since every non singular cubic has genus $pa(3) = (1/2)(2)(1) = 1$.

As another example, $\{y^2 - x^2(x+1) = 0\}$ is singular at $p = (0,0)$ and otherwise is non singular, including at infinity. The partials at the origin are $(-2x - 3x^3, 2y)$ hence they generate the ideal of the power series ring $k[[x,y]]$ at the origin, so this is an odp, $\mu = 1$, and the Riemann surface has genus one less than a non singular cubic, i.e. $1 - 1 = 0$.

Since a line is isomorphic as we shall see later, to P^1 , hence diffeomorphic to a sphere, the union of two transverse lines looks topologically like two spheres joined at one point. There are two branches at the singularity and the Milnor number is 1, since we may take affine coordinates in which the two lines are defined by $xy = 0$, and the singularity is at $(0,0)$. Thus the smooth curve of degree 2 is obtained topologically from two spheres by removing a small neighborhood of their common point, i.e. removing one disc from each sphere, and replacing them by a single connected manifold with boundary consisting of two circles and having the homotopy type of 1 circle. This is a tube, or cylinder, and results in a surface again topologically equivalent to a sphere, so a non singular degree two projective plane curve, i.e. a non singular plane conic, has genus 0.

For a non singular curve of degree 3 we know the genus is one, but we can see it this way also. Let us consider a non singular conic and a line transverse to the conic, meeting it twice again in two ordinary double points. Then both surfaces that meet have genus zero, and they meet at ordinary double points each of Milnor number 1, but there are two of them, so we join the two spheres at two points using two separate cylinders, and we obtain a torus of genus one. Similarly each time we add a transverse line to a non singular curve of degree $d-1$ to get a singular curve of degree d , we obtain $d-1$ ordinary double points at the intersections, and the genus of a nearby smoothed non singular curve of degree d , is increased by $d-2$ from that of the non singular curve of degree $d-1$. Hence the genus of a curve of degree d equals $1 + 2 + \dots + (d-2) = (1/2)(d-1)(d-2)$, and the formula holds for all $d \geq 1$. Now that we have used the genus change from curves with easy singularities to learn the genus of a non singular curve, we can work backwards to learn the genus of some curves with more complicated singularities.

To recap, on a singular curve a small neighborhood of an isolated singularity p with r local analytic branches looks topologically like a cone over a disjoint union of r circles, i.e. a one point union of r discs. [The boundary circles are linked and/or knotted however in the 3 sphere in C^2 .] We know the Riemann surface is then obtained by replacing this neighborhood of the singularity by a disjoint union of r discs. Note that in some sense, these two sets have the same boundary curve, a disjoint union of r circles. Milnor's theory (which is classical in this dimension) tells us how to obtain a topological model of a nearby non singular curve in a related way. I.e. if our singular curve is $f = 0$, a

nearby non singular curve will have form $f = e$ for some small non zero complex number e . A small open ball centered at the original singularity p this time will meet the nearby non singular curve in a set of non singular points, hence an open Riemann surface. Milnor's theory gives us the topology of this Riemann surface. It says the intersection of the nearby non singular curve with a small closed ball centered at p , will be a compact connected real 2 manifold, a “handlebody” whose boundary is still r disjoint circles, and whose 1st betti number equals $\mu = \dim(C[[x,y]]/(\partial f/\partial x, \partial f/\partial y))$, as a finite dimensional complex vector space. Indeed this manifold has the homotopy type of a wedge of μ circles. If we cap off each boundary circle in the handlebody by a disc we get a manifold of genus g' where $g' + (r-1)$ is the difference in the genus of the Riemann surface of the original singular curve and the genus of the nearby non singular curve, assuming p is the only singularity of our singular curve. I.e. $g' + (r-1)$ is the drop in the genus of the non singular curve attributed to that one isolated singular point p . As stated above this drop equals $\partial(p) = (1/2)(\mu + r-1) = g' + (r-1)$, i.e. $\mu = 2g' + r-1$.

Test this on a curve with affine equation $y^2 = x^6 - 1$. This curve is non singular except at infinity, and we find the homogeneous equation $y^2 z^4 = x^6 - z^6$, and set $z = 0$. Then we get $0 = x^6$, a single point of the projective plane $[0:1:0]$. To find an affine equation at this point we set $y = 1$, and get the equation $f(x,z) = z^4 - x^6 + z^6 = 0$. Then $\partial f/\partial x = -6x^5$, and $\partial f/\partial z = 4z^3 + 6z^5 = z^3(4 + 6z^2)$. The ideal these two partials generate in the power series ring $k[[x,z]]$ is generated by (x^5, z^3) since in the power series ring $4 + 6z^2$ is a unit. Thus the dimension μ of the quotient vector space $k[[x,z]]/(\partial f/\partial x, \partial f/\partial z) = k[[x,z]]/(x^5, z^3)$, equals the number of monomials of form $x^i z^j$ with $0 \leq i < 5$, $0 \leq j < 3$, i.e. $\mu = 15$. Now we have studied hyperelliptic curves like this in pictures and we know this one has two local branches at infinity. Thus we obtain a smooth curve of degree 6 from this singular degree 6 curve by plumbing in a connected manifold with two boundary circles, and the homotopy type of a wedge of 15 circles. This is a manifold of genus 7 with 2 discs removed. Whether you can see that or not, our formulas show that plumbing this manifold into the hole left by removing the singularity, raises the genus of our singular curve by $(1/2)(\mu + r-1) = 8$. In fact the formula for the genus of a non singular curve of degree 6 says it should be 10, and we know from our work in class that the singular curve $\{y^2 = x^6 - 1\}$ above, has a Riemann surface which is a double cover of a sphere with 6 branch points, hence has genus 2. So it checks out.