Chapter 2
Portfolio Theory: Origins, Markowitz and CAPM Based Selection

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The valuation of risky assets was initially based on bond valuation theory. Although the valuation of a bond may fluctuate due to variation in market interest rates, the coupon was fixed and subject mainly to the risk of default, which was episodic rather than continuous; prominent in the nature of the instrument were certain legal safeguards. When applied to stocks (risky assets) frequently the role of the coupon rate was played by the dividend, which though not fixed was deemed to be steady and subject only to infrequent changes. This framework, however, is evidently inappropriate in the case of stocks where the rate of return (principally earnings) is inherently variable and is not subject to legally binding specification.

The origin of modern finance in this context (portfolio selection) must be traced to the work of Markowitz (1952, 1956, 1959). Its basic framework is based on the work of von Neumann and Morgenstern (1944) (VNM) who pioneered the view that choice under uncertainty may be based on expected utility. The concept of utility is at least as old as the nineteenth century and the view that consumer choice (of the basket of goods and services consumed) was a compromise between the consumer's desires and the resources available to him (income). Thus, preceding expected utility constructs, the view prevailed that consumers obtained the most preferred bundle of goods and services they could attain with their incomes. But how could we import these concepts into the valuation of risky assets and their subsequent inclusion in a basket we call portfolio; after all consumers choose various goods because they satisfy some desire or group of desires. But a consumer (investor) need not have a preference or desire to own a given security per se. The importance of

I wish to thank John B. Guerard Jr. for helpful comments and stimulation.

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J.B. Guerard, Jr. (ed.), *Portfolio Construction, Measurement, and Efficiency*,
DOI 10.1007/978-3-319-33976-4_2

Markowitz' contribution is that he isolated two aspects of relevance, return and risk, established a method of ranking them (a utility function), thus recognizing the inherent riskiness (randomness) of returns, and invoked VNM in the process. Having done so, it becomes clear that in this formulation the problem is conceptually broadly similar to the problem of consumer choice, although by no means identical. He correctly saw that it is not possible simultaneously to increase returns and at the same time minimize the risk entailed, because of arbitrage. Indeed, many of the later developments of the subject follow from these insights although not explicitly detailed in Markowitz (1959).

2.1 Constrained Optimization

Ignoring the utility or expected utility aspects, the (portfolio) selection problem was defined as: maximize expected returns subject to a variance and scale constraint.[1] Setting up the Lagrangian

$$\Lambda = \gamma' Er + \alpha r_0 + \lambda_1(k - \gamma'\Sigma\gamma) + \lambda_2(1 - e'\gamma - \alpha), \tag{2.1}$$

where E is the expectation operator, r is an n-element column vector containing the rates of return on the risky assets, r_0 is the risk free rate, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)'$ is the portfolio composition, the individual elements γ_i denoting the proportion of the portfolio invested in the ith risky asset and α is the portion invested in the risk free asset; evidently, $\gamma'\Sigma\gamma$ is the variance of the portfolio, or its risk; it is assumed that at least for the duration of the choice period,

$$Er = \mu \qquad \mathrm{Cov}(r) = \Sigma > 0, \quad Er_0 = r_0 \quad \mathrm{var}(r_0) = 0. \tag{2.2}$$

If we solve for the first order conditions we find[2]

$$\gamma = \frac{1}{2\lambda_1}\Sigma^{-1}(\mu - er_0), \quad \alpha = 1 - \frac{1}{2\lambda_1}e'\Sigma^{-1}(\mu - er_0), \tag{2.3}$$

$$\lambda_1 = \frac{\gamma'(\mu - er_0)}{2\gamma'\Sigma\gamma}, \quad \lambda_2 = r_0, \quad e = (1, 1, 1, \ldots 1)'. \tag{2.4}$$

[1]How does one explain that only the mean and variance of returns and not other moments play a role? One can justify this by an implicit assumption that the probability distribution of returns belongs to a family of distributions described by **only two** parameters, or that the expected utility function is of such a form that it depends only on the mean and variance of the relevant distribution.

[2]It should be noted that Markowitz did not actually solve for γ; rather his version focused only on risky assets and imposed **non-negativity constraints on the elements of** γ. Thus what he derived from the first order conditions were rules for inclusion in and/or exclusion from (of securities) in an optimal portfolio.

Although the solution was easy to obtain the interpretation of the Lagrange multiplier, λ_1 is clouded by the fact that **it is not invariant to scale**; thus if we were to double α and the elements of γ, the expression for the Lagrange multiplier would be halved without any change in other aspects of the procedure; thus any interpretation given to it in comparisons would be ambiguous and questionable. To that end we alter the statement of the constraint, thus redefining risk, to[3]

$$k = (\gamma' \Sigma \gamma)^{1/2} = \sigma_p.$$

without changing its substance. In turn this will yield the solution

$$\gamma = \frac{k}{\lambda_1} \Sigma^{-1} (\mu - e r_0), \quad \alpha = 1 - \frac{k}{\lambda_1} e' \Sigma^{-1} (\mu - e r_0), \tag{2.5}$$

$$\lambda_1 = \frac{\gamma'(\mu - e r_0)}{\sigma_p}, \quad \lambda_2 = r_0, \quad e = (1, 1, 1, \ldots 1)'. \tag{2.6}$$

Examining the numerator of λ_1, i.e. **the Lagrange multiplier in the alternative formulation of the risk constraint** we find

$$\gamma'(\mu - e r_0) = (\gamma' \mu + \alpha r_0) - r_0, \tag{2.7}$$

i.e. **it is the excess expected return of the portfolio** while the denominator is σ_p, i.e. the portfolio's risk! Thus the Lagrange multiplier attached to the risk constraint, in the Markowitz formulation, gives us the 'terms of trade' between reward and risk at the optimum. Noting further that

$$\frac{\partial \Lambda}{\partial k} = \lambda_1,$$

[3]From the point of view of computation, entering the constraint as $k^2 = \gamma' \Sigma \gamma$ simplifies operations, but makes the Lagrange multiplier harder to interpret in terms of common usage in finance; if, however, we enter the constraint as $k = (\gamma' \Sigma \gamma)^{1/2}$, we complicate the computations somewhat, we do not change the nature of the solution, but we can interpret the Lagrange multiplier in terms of common usage comfortably. We should also bear in mind that if risk is defined in terms of the standard deviation rather than the variance, a certain intuitive appeal is lost. For example, it is often said that security returns are subject to two risks, market risk and idiosyncratic risk. If we also say, as we typically do, that market risk is independent of idiosyncratic risk, then we have the following situation: denote the market risk by the variance of a certain random variable, say σ^2_{mar} and the idiosyncratic risk by the variance σ^2_{idio} then the risk of the security return is the **sum** $\sigma^2_{\mathrm{mar}} + \sigma^2_{\mathrm{idio}}$. On the other hand, if we **define risk in terms of the standard deviation**, then the two risks are not additive, i.e. the risk of the security is **not** $\sigma_{\mathrm{mar}} + \sigma_{\mathrm{idio}}$ but $\sqrt{\sigma^2_{\mathrm{mar}} + \sigma^2_{\mathrm{idi0}}}$, which is smaller, when we use as usual the positive square root. This problem occurs whenever there is aggregation of independent risks.

we may interpret λ_1 as the optimal marginal reward for risk or more correctly the marginal reward for risk at the optimum. All this is, of course, ex ante and assumes that the investor or the portfolio manager knows with certainty the mean and variance of the stochastic processes that determine ex post the realized returns.

2.2 Portfolio Selection and CAPM

Another aspect that needs to be considered is whether the index based on the interpretation of the Lagrange multiplier discussed in connection with the solution given to the portfolio selection model inv Markowitz (1959) is relevant in the CAPM context and whether these optimality procedures shed any light on the issue of composition rules.

For the latter issue, a more recent development along these lines is given in Elton et al. (2007), where the objective is stated as the maximization of the Sharpe ratio, which is the ratio of (expected) excess returns to (expected) standard deviation of a portfolio, using CAPM as the source of the covariance structure of the securities involved. It does that by means of nonlinear programming; from the first order conditions it derives rules of inclusion in (and exclusion from) an optimal portfolio. While similar in objective, this **is not** equivalent to the Markowitz approach. Moreover, it is questionable that maximizing the Sharpe ratio is an appropriate way for constructing portfolios. In particular, a portfolio consisting of a single near risk free asset with near zero (but positive) risk and a very small return might well dominate, in terms of the Sharpe ratio, any portfolio consisting of risky assets in the traditional sense. A ratio can be large if the numerator is large relative to the denominator, **or if the denominator is exceedingly small relative to a small positive numerator**. Consider (10/2) and (.5/0.1) or (.1/0.01). The point is that **given the level of risk** it is generally agreed that the higher the Sharpe ratio the better, however, to put it mildly, it is not generally accepted that the higher the Sharpe ratio the better, **irrespective of risk**. Evidently this would depend on the investor's or portfolio manager's tradeoff between risk and reward.

In Markowitz the rates of return are stochastic processes with fixed means and covariance matrix; thus what is being solved is an essentially static problem. It could be made somewhat dynamic by allowing these parameters to change over time, perhaps discontinuously.[4] This, however, imposes a considerable computational burden, viz. the re-computation of n means and $n(n+1)/2$ variances and covariances. On the other hand, if we adopt the framework of CAPM suggested, by Sharpe (1964), Lintner (1965b), Mossin (1966), Treynor (1962)[5] and others, as originally

[4] I say 'somewhat dynamic' because we still operate within what used to be called a 'certainty equivalent' environment, in that the underlying randomness is not fully embraced as in option price theory.

[5] The intellectual history of the evolution of CAPM is detailed in the excellent and comprehensive paper by French (2003), which details *inter alia* the important but largely unacknowledged role payed by the unpublished paper Treynor (1962). We cite Lintner (1965a) in the cite both Lintner paper of 1965 in his capital market development.

formulated, rates of returns are assumed to behave as

$$r_{ti} - r_{t0} = \beta_i(r_{mt} - r_{t0}) + u_{ti}, \quad i = 1, 2, \ldots, n. \quad t = 1, 2, \ldots T, \tag{2.8}$$

where r_{ti}, r_{t0}, r_{mt} are, respectively, the rates of return on the ith risky asset, the riskless asset and the market rate of return, β_i is a fixed parameter, at least in the context of the planning period; u_{ti} is, for each i, a sequence of independent identically distributed random variables with mean zero and variance ω_{ii}; moreover u_{ti} and $u_{t'j}$ are mutually independent for every pair (t, t') and (i, j). Notice that if we rewrite the CAPM equation as

$$r_{ti} = (1 - \beta_i)r_{t0} + \beta_i r_{mt} + u_{ti}, \tag{2.9}$$

this version of CAPM seems to assert that individual returns are, on the average, linear combinations (more accurately weighted averages for positive betas) of the risk free and market rates **with fixed weights**. A more popular recent version is

$$r_{ti} = c_i + \beta_i r_{mt} + u_{ti}, \tag{2.10}$$

where now c_i is an **unconstrained parameter**. If we bear in mind that the risk free rate is relatively constant it might appear that the two versions are equivalent. However, when considering applications this is decidedly not so. Some of the differences are

1. If we attempt to apply a (Markowitz) optimization procedure using the first version, the component α of the portfolio devoted to risk free assets cannot be determined and has to be provided *a priori*. This is due to the fact that in this version

   $$Er_p = r_{t0} + \gamma'\beta(\mu_{mt} - r_{t0}),$$

   which is the expected value of the returns on any portfolio $(\gamma,\ \alpha)$, **does not contain $\boldsymbol{\alpha}$** ; since the risk free rate has zero variance and zero covariances with the risky assets, α is not contained in the variance (variability) of the portfolio either. Thus, it cannot possibly be determined by the optimization procedure. With the alternative version, however, we can.
2. Bearing in mind that expected returns and risks are **not known and must be estimated prior to portfolio selection**, if we use the first version to determine an asset's beta we obtain

   $$\hat{\beta}_i = \frac{\sum_{t=1}^{T}(r_{ti} - r_{t0})(r_{mt} - r_{t0})}{\sum_{t=1}^{T}(r_{mt} - r_{t0})^2}, \quad \hat{u}_{ti} = r_{ti} - \hat{\beta}_i(r_{mt} - r_{t0}), \qquad \hat{\omega}_{ii} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{ti}^2;$$

   if we use the alternative (second) formulation of CAPM with an unrestricted constant term we would obtain

$$\tilde{\beta}_i = \frac{\sum_{t=1}^{T}(r_{ti} - \bar{r}_i)(r_{mt} - \bar{r}_m)}{\sum_{t=1}^{T}(r_{mt} - \bar{r}_m)^2}, \quad \tilde{u}_{ti} = r_{ti} - \hat{\beta}_i(r_{mt} - r_{t0}), \quad \tilde{\omega}_{ii} = \frac{1}{T}\sum_{t=1}^{T}\hat{u}_{ti}^2;$$

If the risk free rate is appreciably smaller than the sample means of individual asset and market rates of return, the estimates of β_i could deviate appreciably from those obtained using the first version.

3. Equation (2.11) implies that every individual asset's rate of return is a linear combination of the risk free and market rates **but the coefficients of the linear combination need not be positive**. In particular it implies that an asset with negative beta does not respond to market rates as its beta might indicate, but the response is modulated by the term $(1-\beta)r_{t0}$, which in this case is positive. In addition, it may have implications for well-diversified portfolios that have not yet been explored.

Thus, we shall conduct our analysis on the basis of the alternative (second) version of CAPM given in Eq. (2.12).

The main difference between our formulation and that in Markowitz is that here r_{mt} is a **random** variable with mean μ_{tm} and variance σ_{tm}^2 whose parameters **may vary with t, perhaps discontinuously**; it is, however, independent of $u_{t'i}$, for every pair (t, t') and i; moreover, if we use it as the basis for a Markowitz type procedure the resulting portfolios would depend on these parameters. Thus they could form the basis for explicit dynamic adjustment as their parameters vary in response to different phases in economic activity.

Within each t, the analysis is **conditional** on r_{mt}. The relation may be written, for a planning horizon T,

$$r_{ti} = c_i + \beta_i r_{mt} + u_{ti}, \quad i = 1, 2, \cdots, n \quad t = 1, 2, \cdots, T \tag{2.11}$$

where r_{ti}, r_{mt} are, respectively, the observations on the risk free and market rates at time t, c_i, β_i are parameters to be estimated and u_{ti} the random variables (error terms), often referred to as idiosyncratic risk, with mean zero and variance ω_{ii}. Because the analysis is done **conditionally** on r_{mt} and because **by assumption** the u_{ti} are independently distributed, and all equations contain the same (right hand, explanatory) variables, we can estimate the unknown parameters **one equation at a time without loss of efficiency, by means of least squares**. Now, can we formulate a Markowitz like approach in choosing portfolios on the basis of CAPM? Before we do so it is necessary to address an issue frequently mentioned in the literature, viz. that by diversification we may eliminate 'idiosyncratic risk'. What does that mean? It could simply mean that in a diversified portfolio idiosyncratic risk emanating from any one risky asset or a small class thereof is negligible relative to market risk, although it need not be zero. On the other hand, taken literally it means that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\gamma_i\epsilon_{ti} \overset{\text{a.c.}}{\to} 0, \tag{2.12}$$

i.e. this entity converges to zero with probability 1, and thus idiosyncratic risk need not be taken into account, meaning that for the purpose of portfolio selection we can use a version of CAPM which **does not contain an idiosyncratic risk component**. Formally, what is required of such entity in order to converge (to its zero mean) with probability one? For example, in the special case where $\gamma_i \approx 1/n$, a sufficient condition for Eq. (2.14) to hold is given by Kolmogorov as[6]

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{\omega_{ii}}{i^2}\right) < \infty,$$

which would be satisfied if the ω_{ii} are bounded. For another selection of the components of γ it may not be; for example, if $\gamma_i \approx n^{\eta}/n, \eta > 0$ it will not be satisfied even if the variances are bounded. Since this assertion imposes a restriction on the vector, γ, of an undetermined nature, we prefer to explicitly take into account idiosyncratic risk in formulating the problem of optimal portfolio selection.

Another aspect that needs to be considered is whether the index based on the interpretation of the Lagrange multiplier discussed in connection with the solution given to the portfolio selection model in Markowitz (1959) is relevant in the CAPM context and whether these optimality procedures shed any light on the issue of composition rules.

We proceed basically as before except now the variability constraint utilizes the standard deviation. For clarity, we redefine portfolio returns and the covariance matrix of the securities involved given the CAPM specification; thus

$$r_p = \gamma' c + \gamma' \beta r_{mt} + \alpha r_{t0} + \gamma' u'_{t\cdot}, \quad \Sigma = \Omega + \sigma_{mt}^2 \beta\beta', \tag{2.13}$$

and the solution is obtained by optimizing the Lagrangian

$$\Lambda = \gamma' c + \gamma' \beta r_{mt} + \alpha r_{t0} + \lambda_1 [k - (\gamma' \Sigma \gamma)^{1/2}] + \lambda_2 (1 - \gamma' e - \alpha), \tag{2.14}$$

From the first order conditions we easily obtain

$$\left(\Omega + \sigma_{mt}^2 \beta\beta'\right) \gamma = \frac{k}{\lambda_1} (c + \beta \mu_{tm} - e r_{t0}) \tag{2.15}$$

$$\alpha = 1 - \gamma' e, \quad \lambda_2 = r_{t0}, \quad e = (1, 1, \ldots 1)' \tag{2.16}$$

$$\lambda_1 = \frac{E r_p - r_{t0}}{[\gamma' (\Omega + \sigma_t^2 \beta\beta')^{\gamma}]^{1/2}}. \tag{2.17}$$

[6]See Dhrymes (2013, pp. 202–203).

The last equation is easily obtained by premultiplying the first equation above by γ' and using the definition of Er_p implied by Eq. (2.15) above. If we now substitute for λ_1 we obtain an equation that involves only α and γ, i.e.

$$(\gamma' c + \gamma'\beta\mu_{mt} + \alpha r_{t0})\gamma = k^2 \left(\Omega + \sigma_{mt}^2 \beta\beta'\right)^{-1} (c + \beta\mu_{mt} - er_{t0}). \tag{2.18}$$

But, if we use Eq. (2.16) we can eliminate α so that Eq. (2.18) may be rewritten as

$$\gamma\gamma'(c + \beta\mu_{mt} - er_{t0}) - \gamma r_{t0} = k^2 \left(\Omega + \sigma_{mt}^2 \beta\beta'\right)^{-1} (c + \beta\mu_{mt} - er_{t0}), \tag{2.19}$$

which can now be solved for γ.

A number of features of this procedure need to be pointed out:

1. No high dimensional matrix needs to be inverted, due to a result (Corollary 2.5),[7] which enables us to write

$$\left(\Omega + \sigma_{mt}^2 \beta\beta'\right)^{-1} = \Omega^{-1} - \zeta\Omega^{-1}\beta\beta'\Omega^{-1}, \quad \zeta = \frac{\sigma_{mt}^2}{1 + \sigma_{mt}^2 \beta'\Omega^{-1}\beta};$$

   since Ω **is diagonal** we easily compute

$$\beta'\Omega^{-1}\beta = \sum_{i=1}^{n} \left(\frac{\beta_i^2}{\omega_{ii}}\right), \quad \Omega^{-1}\beta\beta'\Omega^{-1} = \left[\frac{\beta_i\beta_j}{\omega_{ii}^2}\right],$$

   i.e., it is a matrix whose typical element is $\beta_i\beta_j/\omega_{ii}^2$.
2. The number of parameters that we need to estimate prior to optimization is $3n+2$, viz. the elements of the vectors c, β and the variances ω_{ii}; all of these can be obtained from the output of n simple regressions. The other two parameters are simply the mean and variance of the market rate.
3. The procedure yields a set of equations which are quadratic in γ; the solution is a function of k^2, μ_{mt}, σ_{mt}^2 and can be adjusted relatively easily when updating of the estimates of μ_{mt}, σ_{mt}^2 is deemed appropriate.
4. It is interesting that the optimal (solution vector) composition vector, γ, is a **function of (depends on) the risk parameter** k^2, **not** k, i.e. **risk is represented by the variance, not the standard deviation**.

We thus see that in the context of CAPM the implementation of optimal portfolio selection becomes much simpler and computationally more manageable and, consequently, so is the task of evaluation *ex post*.

[7] See Dhrymes (2013, pp. 46–47).

2.3 Conclusion

In this paper we reconsidered the problem of portfolio selection as formulated by Markowitz (1959) and proposed an extension based on CAPM. This extension highlights certain aspects that represent a considerable simplification; it illuminates issues regarding the estimation of securities betas, the role played by idiosyncratic risk and leads to the formulation of a set of quadratic equations that define the optimal composition of efficient portfolios (the elements of the vector γ), as a function of the selected level of risk and estimates of (expected) market rate and its risk (variance). The only remaining problem is to find an algorithm that solves sets of quadratic equations. This should not be very difficult. Given that, it offers a systematic way in which portfolio managers might insert into the process their evolving views of market rates and their associated risk, when updating is deemed necessary.

An interesting by-product is the potential provided by this framework in evaluating (managed) portfolio performance. In a now classic paper Sharpe (1966) evaluates mutual fund performance by considering realized rates of return for a number of mutual funds over a number of years and computes the standard deviation of such returns. The evaluation relies on the ratio of average returns to their standard deviation. Strictly speaking, these two measures do not estimate ‘constant parameters’ since the composition of the fund is likely to have changed appreciably over the period; thus their ratio is not a ranking of the fund itself. It is, however, **a ranking of the fund cum manager**.

If we use the framework presented in the paper which is based on CAPM we could, in principle, during each period compute from published data the portfolio or fund risk as $\gamma'(\Omega + \sigma_m^2\beta\beta')\gamma$. Thus, the evaluator will have for each period, **both realized returns and risk**. This would make a more satisfactory basis for evaluation.

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Springer
http://www.springer.com/978-3-319-33974-0

Portfolio Construction, Measurement, and Efficiency
Essays in Honor of Jack Treynor
Guerard, J. (Ed.)
2017, XXXIII, 453 p. 56 illus., 49 illus. in color.,
Hardcover
ISBN: 978-3-319-33974-0