Problem 4.
The fundamental rep of $SU(2)$, $D^{(\frac{1}{2})}$, is the action of a 2x2 unitary matrix of determinant 1 on a complex 2-vector(spinor). In index notation it looks like

$$\psi_\alpha \to \psi'_\alpha = N_\alpha{}^\beta \psi_\beta \ , \quad N \in SU(2) \ , \quad \alpha, \beta = 1, 2 \ .$$

We want to examine the action of $SU(2)$ on the product space $\mathbb{C}^2 \otimes \mathbb{C}^2$, spanned by $T_{\alpha\beta} = \sum_i \psi_\alpha^{(i)} \phi_\beta^{(i)}$. Since there is only one antisymmetric 2x2 matrix (up to normalisation) we have the important identity

$$T_{\alpha\beta} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha}) + \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha}) = T_{(\alpha\beta)} + T_{[\alpha\beta]} = T_{(\alpha\beta)} + \frac{1}{2}\varepsilon_{\alpha\beta} T^\gamma{}_\gamma$$

where the bracketed indices represent normalised symmetrisation and antisymmetrisation respectively[3]. Now if T transforms under $D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})}$ then, using $N_\alpha{}^{\alpha'} N_\beta{}^{\beta'} \varepsilon_{\alpha'\beta'} = \varepsilon_{\alpha\beta} \det N = \varepsilon_{\alpha\beta}$, we see that the above decomposition is invariant under the group action,

$$T_{\alpha\beta} \to T'_{\alpha\beta} = N_\alpha{}^{\alpha'} N_\beta{}^{\beta'} T_{\alpha'\beta'} = N_\alpha{}^{\alpha'} N_\beta{}^{\beta'} T_{(\alpha'\beta')} + \frac{1}{2}\varepsilon_{\alpha\beta} T^\gamma{}_\gamma \ .$$

So we get the decomposition

$$D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} = D^{(1)} \oplus D^{(0)} \ .$$

Now let T, S be symmetric spin-tensors transforming under $D^{(1)}$, ie $T_{\alpha\beta} = T_{(\alpha\beta)}$ and similarly for S. First we symmetrise the last three indices in $T_{\alpha\beta} S_{\gamma\delta}$

$$\begin{aligned}
T_{\alpha\beta} S_{\gamma\delta} &= 3 \times \frac{1}{3} T_{\alpha\beta} S_{\gamma\delta} + \frac{1}{3}\left(T_{\alpha\gamma} S_{\beta\delta} - T_{\alpha\gamma} S_{\beta\delta} + T_{\alpha\delta} S_{\beta\gamma} - T_{\alpha\delta} S_{\beta\gamma}\right) \\
&= T_{\alpha(\beta} S_{\gamma\delta)} + \frac{1}{3}\left(T_{\alpha[\beta} S_{\gamma]\delta} + T_{\alpha[\beta} S_{\delta]\gamma}\right) \\
&= T_{\alpha(\beta} S_{\gamma\delta)} + K_{\alpha\delta}\varepsilon_{\beta\gamma} + K_{\alpha\gamma}\varepsilon_{\beta\delta} \ , \quad K_{\alpha\beta} = \frac{1}{6}\varepsilon^{\delta\gamma} T_{\delta\alpha} S_{\beta\delta}
\end{aligned}$$

Then we symmetrise the indices in the K's and the final index in $T_{\alpha\beta} S_{\gamma\delta}$:

$$T_{\alpha\beta} S_{\gamma\delta} = T_{(\alpha\beta} S_{\gamma\delta)} + K_{(\beta\delta)}\varepsilon_{\alpha\gamma} + K_{(\beta\gamma)}\varepsilon_{\alpha\delta} + K_{(\alpha\delta)}\varepsilon_{\beta\gamma} + K_{(\alpha\gamma)}\varepsilon_{\beta\delta} + \frac{1}{2}(\varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma} + \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta})\mathrm{tr}K.$$

All of the $K_{(..)}$ terms are nescessary to maintain the original symmetries of the RHS (and thus can be restored from a single term by imposing said symmetries). Note that all of the terms in the decomposition are individually invariant and irreducible under the action of SU(2). This decomposition can be carried out for all of the terms in $\sum_i T_{\alpha\beta}^{(i)} S_{\gamma\delta}^{(i)}$, thus we have

$$D^{(1)} \otimes D^{(1)} = D^{(2)} \oplus D^{(1)} \oplus D^{(0)} \ .$$

It's worth checking the dimensions of the two decompositions:

$$\dim(D^{(m/2)}) = 2m + 1 \quad \text{so we have} \quad 2 \times 2 = 3 + 1 \ , \quad \text{and} \quad 3 \times 3 = 5 + 3 + 1 \ .$$

[3]Note that $T^\gamma{}_\gamma \neq \mathrm{tr}T$, since the spinor indices are raised and lowered using the SU(2) metric, the ε-tensor. Rather we have $T^\gamma{}_\gamma = \varepsilon^{\gamma\delta} T_{\delta\gamma} = \varepsilon^{12}(T_{12} - T_{21})$.

In terms of quantum mechanics the decompositions imply that a composite system of two spin-half particles has two eigenvalues of the total squared angular momentum operator ($J^2 \sim 2j+1$), 1 and 3, (ie a singlet and a triplet state) and thus there are 4 spin states allowed: the triplet can be either $-1 = (\downarrow\downarrow)$, $0 = \frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow)$ or $1 = (\uparrow\uparrow)$ whilst the singlet is only allows spin $0 = \frac{1}{\sqrt{2}}(\downarrow\uparrow - \uparrow\downarrow)$. Similarly, the composite system of two spin-one particles has three eigenvalues of the total squared angular momentum operator 1, 3 and 5, allowing total spins ranging from -2 to +2 (with appropriate multiplicities) in integer steps.

(Note that the spins in these decompositions always occur in integer steps since you have to contract two indices at a time.)

Problem 5.

Let $\mathcal{V}$ be the space of holomorphic functions $f(\zeta)$ over $\mathbb{C}^2$, parametrised by complex row-vectors ζ^α with $\alpha = 1, 2$.

(a) Define a map

$$g \mapsto T(g) \ , \quad \Big(T(g)f\Big)(\zeta) := (f \circ R_g)(\zeta) = f(\zeta g) \ , \quad g \in SU(2) \ ,$$

(R_g is right multiplication by g). Repeated application gives

$$\Big(T(h)\big(T(g)f\big)\Big)(\zeta) = \big(T(g)f\big)(\zeta h) = f(\zeta h g) = \big(T(hg)f\big)(\zeta) \ .$$

Thus T forms a representation of $SU(2)$. The vector space this rep acts on is $\mathcal{V}$, which is infinite dimensional.

(b) Define an inner product on $\mathcal{V}$,

$$\langle g|f\rangle := \int d^2\zeta d^2\bar{\zeta}\, e^{-\zeta^\dagger\zeta}\overline{g(\zeta)}f(\zeta)$$

under the action of T this becomes

$$\langle T(h)g|T(h)f\rangle = \int d^2\zeta d^2\bar{\zeta}\, e^{-\zeta^\dagger\zeta}\overline{(T(h)g)(\zeta)}(T(h)f)(\zeta) = \int d^2\zeta d^2\bar{\zeta}\, e^{-\zeta^\dagger\zeta}\overline{g(\zeta h)}f(\zeta h) \ .$$

Now shift the integration, $\xi = \zeta h$, and note that both $\zeta^\dagger\zeta$ and the measure $d^2\zeta d^2\bar{\zeta}$ are $SU(2)$ invariants, ie

$$\begin{aligned} &\zeta^\dagger\zeta = \zeta^\alpha\bar{\zeta}_\alpha = \xi^\beta(h^\dagger)_\beta{}^\alpha(\xi^\gamma(h^\dagger)_\gamma{}^\alpha)^\dagger = \xi^\beta(h^\dagger)_\beta{}^\alpha h_\alpha{}^\gamma\bar{\xi}_\gamma = \xi^\dagger\xi \\ \text{and} \quad &d^2\xi = d\xi^1 d\xi^2 = d(\zeta^\alpha h_\alpha{}^1)d(\zeta^\alpha h_\alpha{}^2) = d\zeta^1 d\zeta^2(h_1{}^1 h_2{}^2 - h_2{}^1 h_1{}^2) = d^2\zeta \ , \end{aligned}$$

since $\det h$=1. Sim $d^2\bar{\xi} = \det h^\dagger d^2\bar{\zeta} = d^2\bar{\zeta}$. This gives

$$\langle T(h)g|T(h)f\rangle = \int d^2\xi d^2\bar{\xi}\, e^{-\xi^\dagger\xi}\overline{g(\xi)}f(\xi) = \langle g|f\rangle \ ,$$

so T is a unitary rep of $SU(2)$ under the above inner product.

(c) Let $\mathcal{V}^{(n)} \subset \mathcal{V}$ be the space of homogeneous polynomials of order n, ie

$$\mathcal{V}^{(n)} = \left\{ f(\zeta) \in \mathcal{V} \ \mid \ f(\zeta) = \Upsilon_{\alpha_1 \ldots \alpha_n}\zeta^{\alpha_1}\cdots\zeta^{\alpha_n} \sim \sum_{i=0}^{n} v_i \ (\zeta^1)^i(\zeta^2)^{n-i} \right\}$$

note that the symbols $\Upsilon_{\alpha_1\dots\alpha_n}$ are totally symmetric. Let $f \in \mathcal{V}^{(n)}$, then

$$\begin{aligned}\big(T(g)f\big)(\zeta) = f(\zeta g) &= \Upsilon_{\alpha_1\dots\alpha_n}(\zeta^{\alpha'_1} g_{\alpha'_1}^{\ \alpha_1}) \cdots (\zeta^{\alpha'_n} g_{\alpha'_n}^{\ \alpha_n}) \\ &= \big(g_{\alpha'_1}^{\ \alpha_1} \cdots g_{\alpha'_n}^{\ \alpha_n} \Upsilon_{\alpha_1\dots\alpha_n}\big)\zeta^{\alpha'_1} \cdots \zeta^{\alpha'_n} = \Upsilon'_{\alpha_1\dots\alpha_n}\zeta^{\alpha_1}\cdots\zeta^{\alpha_n}\end{aligned}$$

So $\mathcal{V}^{(n)}$ is invariant under T.

(d) In the above equation we can see that the action of T on $\mathcal{V}^{(n)}$ is equivalent to the canonical action of $SU(2)$ on a symmetric, rank-n tensor, ie $T|_{\mathcal{V}^{(n)}} \simeq D^{(\frac{n}{2})}$. More precisely, we can define a map from the space of symmetric tensors to the space of homogeneous polynomials

$$W : S^{(n)}(\mathbb{C}) \to \mathcal{V}^{(n)} \ : \ \psi_{\alpha_1\dots\alpha_n} \mapsto \psi_{\alpha_1\dots\alpha_n}\zeta^{\alpha_1}\cdots\zeta^{\alpha_n}$$

so that for any rank-n symmetric tensor $\psi = \psi_{\alpha_1\dots\alpha_n}$ we have

$$T(g)W\psi = WD^{(\frac{n}{2})}\psi \ .$$

Note, that for matrix representations, the existence of such an intertwining map simply implies the matrices are similar, ie $T \sim D \iff T = WDW^{-1}$.

(e) To find the generators of the $SU(2)$-rep acting on $\mathcal{V}^{(n)}$ we proceed in the standard manner: Let $g(t)$ be a curve in $SU(2)$ passing through the unit at $t = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(T(g(t))f\big)(\zeta)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(\zeta\, g(t))\big|_{t=0} = \frac{\partial f(\zeta)}{\partial \zeta^\alpha}\big(\zeta\dot{g}(0)\big)^\alpha = \big(\zeta\dot{g}(0)\partial_\zeta\big)f(\zeta) = \mathcal{T}(\dot{g}(0))f(\zeta)$$

Since $\dot{g}(0) \in su(2)$ we can choose the 'standard' basis for the generators $g_i = s_i$ so that $\mathcal{T}(g_i) = \zeta\sigma_i\partial_\zeta$, they obey the standard $su(2)$ commutation rules (as they should!)

$$[\mathcal{T}(g_i), \mathcal{T}(g_j)] = [\zeta\sigma_i\partial_\zeta, \zeta\sigma_j\partial_\zeta] = \zeta[\sigma_i, \sigma_j]\partial_\zeta = 2\mathrm{i}\varepsilon_{ijk}\mathcal{T}(g_k) \ .$$

Now, if we act on $f \in \mathcal{V}^{(n)}$ we get

$$\mathcal{T}(g_i)f(\zeta) = \zeta\sigma_i\partial_\zeta\left(\Upsilon_{\alpha_1\dots\alpha_n}\zeta^{\alpha_1}\cdots\zeta^{\alpha_n}\right) = \sum_k \Upsilon_{\alpha_1\dots\alpha_n}(\zeta\sigma_i)^{\alpha_k}\zeta^{\alpha_1}\cdots\widehat{\zeta^{\alpha_k}}\cdots\zeta^{\alpha_n} \ ,$$

where the 'hat' means that you exclude that term.

Problem 6.

(a)
$$G = \left\{ \mathrm{Mat}(5,\mathbb{R}) \ni D(\Lambda, a) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix} \;\Big|\; \Lambda \in O(3,1)\ , \quad a \in \mathbb{R}^3 \right\}$$

That G is a group under matrix multiplication follows directly from the fact that $O(3,1)$ is a group;

- group product = matrix product and so it is associative
- the product is closed: $D(\Lambda_2, a_2)D(\Lambda_1, a_1) = D(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$
- the unit is in G: $e = D(\mathbb{1}, 1) \in G$
- the inverse is in G: $D(\Lambda^{-1}, -\Lambda^{-1}a)D(\Lambda, a) = D(\mathbb{1}, 1)$

The group product above shows that $D : IO(3,1) \to G$ is a group homomorphism. Since D is obviously bijective, it is actually an isomorphism.

(b) The representation D acts on $\mathbb{R}^5$ as

$$D(\Lambda, a)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Lambda x + ay \\ y \end{pmatrix}$$

For any $\rho \in \mathbb{R}$ we can decompose $\mathbb{R}^5$ as $\quad \mathbb{R}^5 = V_\rho \oplus V_1\ , \quad V_\rho = (\mathbb{R}^4, \rho)\ , \quad V_1 = (0, \mathbb{R})$. Now V_ρ is invariant under G, so D is a reducible representation, but V_1 is not invariant, so D is not completely reducible.

(c)
$$O(3,1) = \left\{ \Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T \eta \Lambda = \eta\ , \quad \eta = \mathrm{diag}(-1,1,1,1) \right\}$$

Let $\Lambda(t)$ parametrise a curve in $O(3,1)$ passing through the unit element at $t = 0$, then

$$0 = \frac{d}{dt}\Lambda(t)^T \eta \Lambda(t)\Big|_{t=0} = \dot{\Lambda}^T(0)\eta + \eta\dot{\Lambda}(0)\ .$$

So the Lie algebra of $O(3,1)$ is

$$so(3,1) = \left\{ \omega \in \mathrm{Mat}(4,\mathbb{R}) \mid \omega^T \eta + \eta\omega = 0 \right\}$$

where we've written the "s" for special because (using linearity and cyclicity of trace) $\mathrm{tr}\omega = \frac{1}{2}\mathrm{tr}(\omega^{\mathrm{T}} + \omega) = \frac{1}{2}\mathrm{tr}(\eta\omega^T\eta + \omega) = 0$, implying its exponentiation always has unit determinant. In components, an element, $\omega = \eta\lambda$ of $so(3,1)$ satisfies $\lambda_{\mu\nu} + \lambda_{\nu\mu} = 0$, so it simply consists of antisymmetric[4] 4×4 matrices. Thus has dimension six.

[4]Note that the symmetry of a linear map (matrix) can only be discussed in combination with a metric, so that there is a natrual bijection between linear maps (one upper and one lower index) and bilinear forms (two lowered indices). A bilinear form maps two vectors into a scalar, and thus you can discuss the symmetry properties with respect to its two arguements. Normally when discussing matrices the standard, Euclidean metric is assumed and indices are raised and lowered using the Kronecker delta, so that there is no difference between the matrix considered as a linear map or a bilinear form. In the problem at hand, the metric used is that of Minkowski, ie η and thus it matters whether an index is upper or lower.