A Mathematical Structure Isomorphic to Reality

version 5-12

3-Oct-2010

Abstract

In this document, the author presents a structure $\mathcal{U}$ with the property that all structures are "elementarily embeddable" within it regardless of underlying symbol set. The motivation is given by *The Mathematical Universe* article by Max Tegmark, [6] and *The Multiverse Hierarchy*, [7], in which it is hypothesized that reality is *a* mathematical structure, left unspecified in both [6] and [7]. Consequently, it is hypothesized that the structure with the aforementioned property could be central in the Mathematical Universe Hypothesis as being the structure isomorphic to reality.

We will find a structure such that *every* structure is elementarily embeddable within it. To that end, we will use a set theory with a universal set, New Foundations with Urelements, NFU, which has been shown to be consistent [9, section 7].

To accomplish our goals, some new definitions of isomorphism and what it means to be elementarily embedded are presented in a way compatible with working with differing symbol sets.

Part II includes a summary of relevant concepts of logic and NFU set theory necessary to reach our goal. This is included so that this document is accessible to a wider audience and may be skipped by readers familiar with these subjects. Since the intended audience of this article includes people not versed in mathematical logic, several elementary concepts are discussed.

Part I
Discussion

In [6], Tegmark discusses two hypotheses: the external reality hypothesis (ERH) and the mathematical universe hypothesis (MUH). To quote Tegmark, the ERH is that "there exists an external physical reality completely independent of us humans" and the MUH is that "our external physical reality is a mathematical structure." He argues that the ERH implies the MUH:

- A complete description of the external reality is called a TOE (theory of everything).
- The ERH implies that for a description to be complete, it must be well-defined also according to non-human sentient entities (say aliens or future supercomputers) that lack the common understanding of concepts that we humans have evolved, e.g., "particle", "observation" or indeed any other

English words. Put differently, such a description must be expressible in a form that is devoid of human "baggage".

- The ERH implies that a "theory of everything" has no baggage.
- Something that has a baggage-free description is precisely a mathematical structure. There are many equivalent ways of describing the same structure, and a particular mathematical structure can be defined as an equivalence class of descriptions. Thus, although any one description involves some degree of arbitrariness (in notation, etc.), there is nothing arbitrary about the mathematical structure itself.
- Therefore, the external physical reality described by the TOE is a mathematical structure.

Notice we can dispense with the words "external" and "physical" and a similar argument shows that the first statement below implies the second:

(premise) **"there exists a reality completely independent of us humans"**

(conclusion) **"reality is a mathematical structure."**

- A complete description of reality is called a TOE.
- The premise implies that for a description to be complete, it must be expressible in a form that is devoid of human "baggage".
- The premise plus the previous step implies that a TOE has no baggage.
- Something that has a baggage-free description is precisely a mathematical structure.
- Therefore, reality described by the TOE is a mathematical structure.

This premise is intended to be less controversial than the ERH but leading to the same conclusion about a complete description of reality.

What is meant by the expression "mathematical structure?" [7] provides an intuitive explanation of what a mathematical structure is by saying, "all mathematical structures are just special cases of one and the same thing: so-called formal systems. A formal system consists of abstract symbols and rules for manipulating them, specifying how new strings of symbols referred to as theorems can be derived from given ones referred to as axioms."

A precise definition of mathematical structure will be discussed part II. Regarding that precise definition, the set of abstract symbols corresponds to a symbol set S which consists of constant symbols, function symbols, and relation symbols. The rules for manipulating those symbols is given by what the author calls structure maps which assign either a constant, a function, or a relation to the symbols in S. These structure maps are rules for the interpretation of the symbols in S.

In [7], Tegmark argues for there being no more than four types of parallel universes. Moreover, he writes that the so-called Level IV multiverse employs a sort of "mathematical democracy," meaning that parallels governed by other equations are equally real. Furthermore, we argued that reality is a mathematical structure. **This document means to progress towards an answer to the question, "which mathematical structure is isomorphic (and hence equivalent) to reality?"** Tegmark's most recent article on the subject, [7], states "the true mathematical structure isomorphic to our universe, if it exists, has not yet been found."

We will investigate a structure $\mathcal{U}$ with the property that all structures are elementarily embeddable within it.

If structure $\mathcal{A}_1$ is elementarily embedded within structure $\mathcal{A}_2$ via an injective map, then $\mathcal{A}_1$ is logically equivalent to a substructure of $\mathcal{A}_2$. Therefore, one could in some sense almost be correct in saying that $\mathcal{A}_1$ *is* a substructure of $\mathcal{A}_2$ because $\mathcal{A}_1$ is logically indistinguishable from a substructure of $\mathcal{A}_2$. Consequently, a structure in which all structures are elementarily embeddable within it would entail that structure is, in a sense, the ultimate mathematical structure.

The author's hypothesis is that a structure with the property that all structures are elementarily embeddable within it is (isomorphic to) reality.

Here is an outline of the steps towards finding such a structure:

1. Show that the set of all structures exists. The stratified comprehension theorem borrowed NFU set theory enables us to show this set exists by specifying a first-order formula that defines the statement, "x is an $S-$structure," where S is a symbol set.

2. Form a kind of product of all structures, called a reduced product, which is a generalization of an ultraproduct. The set found in step one will be the index set for this product *and* the family of structures used to form the product.

3. It will be shown that every structure in the trivial reduced product $\mathcal{U}$ is elementarily embeddable within the product. Therefore, the product of all structures has the property that all structures are elementarily embeddable within it.

Part II
Background Concepts

1 Tools from NFU Set Theory

In this document, all sets are those consistent with NFU set theory. A textbook on NFU, which is Quine's ***N***ew ***F***oundations with extensionality weakened to allow ***u***relements, is presented in [4]. Please note that NFU has been shown to be consistent [see 9, section 7].

The tools mentioned in this section used in **Part III** form the foundation for defining a structure with the property that all structures are elementarily embeddable within it. For the purposes of this document, the most important of these is the theorem of stratified comprehension which allows us to say a wide variety of things are sets, such as the set of all structures with respect to a language.

Axiom of the Universal Set. The set $\{x : x = x\}$, also called V, exists.

A set of ordered n-tuples is called a **relation.** A **binary relation** is a set of ordered pairs. If R is a binary relation, $(x, y) \in R$ is often denoted xRy.

Axiom of Domains. If R is a binary relation, the set

$$\mathsf{dom}\,(R) = \{x : \exists y\,(xRy)\}\,,$$

called the domain of R, exists.

Axiom of Inclusion. The set $[\subseteq] := \{(x, y) : x \subseteq y\}$ exists.

Theorem $[\in] := \{(x, y) : x \in y\}$ does not exist.

A proof of this theorem, proved in the spirit of Russell's Paradox, can be found in [4].

Axiom of Projections. The following sets exist:

$$\pi_1 = \{((x, y)\,, x) : x, y \in V\}$$

and

$$\pi_2 = \{((x, y)\,, y) : x, y \in V\}\,.$$

The following stratified comprehension theorem will prove to be an invaluable tool later on. In order to discuss the theorem of stratified comprehension, we first define a stratified formula.

Definition A formula ϕ of first-order logic involving no relation other than $\in$, π_1, π_2, or $=$ is said to be **stratified** if it is possible to assign a non-negative integer to each variable x in ϕ, called the **type** of x, in such a way that

(a) Each variable has the same type wherever it appears.

(b) In each atomic formula $x = y$, $x\pi_1 y$, $x\pi_2 y$ in ϕ, the types of the variables x and y are the same.

(c) In each atomic formula $x \in y$ in ϕ, the type of y is one higher than the type of x.

Stratified Comprehension Theorem For each stratified formula ϕ, the set $\{x : \phi\}$ exists.

This is proved in [4]. Note that the formula $\mathsf{not}\,(x \in x)$ used in Russell's Paradox is not stratified because parts (a) and (c) can not simultaneously be satisfied.

Corollary $[x \to y]$ which is defined to be the set of all functions from x to y, exists by the stratified comprehension theorem.

Axiom of Singletons For every object x, the set $\{x\} = \{y : y = x\}$ exists, and is called the singleton of x.

The **power set** of a set x, denoted $\mathcal{P}(x)$, is the set of all subsets of x:

$$\mathcal{P}(x) := \mathsf{dom}\left([\subseteq] \cap V \times \{x\}\right).$$

This set exists by the axioms of domains, inclusion, universal set, and singletons.

2 Elements of First-Order Logic

This section is a summary of topics in logic that will serve us later on. Most the material in this section is drawn from [2] and [3] in addition to some new material which involves a study of structures with differing symbol sets.

2.1 Symbol Sets

Symbol sets consist of the following:

1. a (possibly empty) set of *constant symbols.*
2. For every $k \geq 1$ a (possibly empty) set of k-ary *function symbols*;
3. For every $k \geq 1$ a (possibly empty) set of k-ary *relation symbols.*

A **k-ary function on** A is any function from $A^k \to A$ and **k-ary relation on a set** A is any subset of A^k.

2.2 Structures

A pair (A, α) is a **structure** of a symbol set S if and only if A is a set and α is a map whose domain is S with the following properties:

1. for every constant symbol $c \in S$, $\alpha(c) \in A$,
2. for every k-ary function symbol $f \in S$, $\alpha(f)$ is a k-ary function on A, and
3. for every k-ary relation symbol $R \in S$, $\alpha(R)$ is a k-ary relation on A.

Such a structure is called an S-structure.

For example, let A be the set of integers and $S = \{z, p, l\}$ a symbol set where z is a constant symbol, p is a binary function symbol, and l is a binary relation symbol. Define a map α so that $\alpha(z) = 0$, $\alpha(p)$ is the usual plus operation, and $\alpha(l)$ is the usual less than relation. Then (A, α) is an S-structure.

Script letters will denote structures and if $\mathcal{A} = (A, \alpha)$ is a structure, A is called the **universe** (or **domain**) of $\mathcal{A}$. The author will call such an α a **structure map**. For all constant symbols $c \in S$, $\alpha(c)$ is denoted $c^{\mathcal{A}}$, for all function symbols $f \in S$, $\alpha(f)$ is denoted $f^{\mathcal{A}}$, and for all relation symbols $R \in S$, $\alpha(R)$ is denoted $R^{\mathcal{A}}$.

2.3 Terms

Given a symbol set S, τ is an $S-$**term** if one of the following conditions hold:

1. τ is a constant;
2. τ is a variable; or
3. there is a k-ary function symbol $f \in S$ and $S-$terms $\tau_1,...,\tau_k$ such that τ is $f\tau_1...\tau_k$.

2.4 Atomic Formulas

Intuitively, atomic formulas are meant to represent statements that cannot be broken down any further using connectives and quantifiers.

More precisely, we say that ϕ is an **atomic** $S-$**formula** if either

1. there is a k-ary relation symbol $R \in S$ and there are $S-$terms $\tau_1,...,\tau_k$ such that ϕ is $R\tau_1...\tau_k$ or
2. there are $S-$terms τ_1 and τ_2 such that ϕ is $\tau_1 = \tau_2$.

2.5 Non-atomic Formulas, Logical Connectives, and Quantifiers

The following are called logical connectives:

connective	meaning
$\neg$	not
$\wedge$	and
$\vee$	or
$\rightarrow$	conditional
$\leftrightarrow$	biconditional
$=$	equality

Also, there are two quantifiers $\exists$ and $\forall$ meaning "there exists" and "for all," respectively.

A generic $S-$**formula** is defined inductively as follows:

- If ϕ is an $S-$formula, then so is $\neg\phi$;

- If ϕ_1 and ϕ_2 are $S-$formulas, then so is $\phi_1 \odot \phi_2$ where $\odot$ could be any of the connectives other than the negation symbol;
- If ϕ is an $S-$formula and x is a variable, then $\exists x\phi$ is an $S-$formula;
- If ϕ is an $S-$formula and x is a variable, then $\forall x\phi$ is an $S-$formula; and
- ϕ is an $S-$formula only if either it is an atomic $S-$formula or a formula obtained by finitely many applications of the above rules.

Note that some of these symbols can be defined in terms of $\neg$, $\vee$, and $\exists$ which will make future definitions and induction proofs more compact. Let ϕ_1 and ϕ_2 be $S-$formulas and x a variable.

logical symbol	interpretation
$\phi_1 \wedge \phi_2$	$\neg(\neg\phi_1 \vee \neg\phi_2)$
$\phi_1 \to \phi_2$	$\neg\phi_1 \vee \phi_2$
$\phi_1 \leftrightarrow \phi_2$	$\neg(\neg(\neg\phi_1 \vee \phi_2) \vee \neg(\phi_1 \vee \neg\phi_2))$
$\forall x\phi_1$	$\neg\exists x\neg\phi_1$

2.6 Induction Principle for Formulas

In order to show that all S-formulas have a certain property P, it is sufficient to show:

1. Every atomic S-formula has the property P.
2. If the S-formula ϕ has the property P, then $\neg\phi$ also has property P.
3. If ϕ_1 and ϕ_2 are S-formulas that have property P, then $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, $\phi_1 \to \phi_2$, and $\phi_1 \leftrightarrow \phi_2$ also have property P.
4. If the S-formula ϕ has the property P and if x is a variable, then $\forall x\phi$ and $\exists x\phi$ also have property P.

These four steps intuitively follow the inductive definition of formulas but we can actually save ourselves some trouble by using the fact that $\wedge$, $\to$, $\leftrightarrow$, and $\forall$ can be eliminated; therefore to prove that all S-formulas have a property P, it is sufficient to show:

1. Every atomic S-formula has the property P.
2. If the S-formula ϕ has the property P, then $\neg\phi$ also has property P.
3. If ϕ_1 and ϕ_2 are S-formulas that have property P, $\phi_1 \vee \phi_2$ also has property P.
4. If the S-formula ϕ has the property P and if x is a variable, then $\exists x\phi$ also has property P.

2.7 Sentences and n-Formulas

The set of variables, var_S, in an S-term is defined inductively where x is a variable, c is a constant, f is a k-ary function, and τ_1,..., τ_k are terms:

$$\text{var}_S(x) := x$$

$$\text{var}_S(c) := \emptyset$$

$$\text{var}_S(f\tau_1 \ldots \tau_k) := \bigcup_{1 \leq j \leq k} \text{var}_S(\tau_j).$$

Fix a symbol set S. The set of **free variables** in a formula ϕ, denoted by $\text{free}(\phi)$ is defined inductively where τ_1,..., τ_k are terms, and R is a k-ary relation:

$$\text{free}(\tau_1 = \tau_2) := \text{var}_S(\tau_1) \cup \text{var}_S(\tau_2)$$

$$\text{free}(R\tau_1 \ldots \tau_k) := \bigcup_{1 \leq j \leq k} \text{var}_S(\tau_j)$$

$$\text{free}(\neg\phi) := \text{free}(\phi)$$

$$\text{free}(\phi_1 \vee \phi_2) := \text{free}(\phi_1) \cup \text{free}(\phi_2)$$

$$\text{free}(\exists x\phi) := \text{free}(\phi) - \{x\}.$$

Due to the ability to reduce the necessary list of connectives and quantifiers down to just $\neg$, $\vee$, and $\exists$, this definition effectively applies to all formulas. Consequently, the following are true:

$$\text{free}(\phi_1 \wedge \phi_2) = \text{free}(\phi_1) \cup \text{free}(\phi_2)$$

$$\text{free}(\phi_1 \to \phi_2) = \text{free}(\phi_1) \cup \text{free}(\phi_2)$$

$$\text{free}(\phi_1 \leftrightarrow \phi_2) = \text{free}(\phi_1) \cup \text{free}(\phi_2)$$

$$\text{free}(\forall x\phi) = \text{free}(\phi) - \{x\}.$$

A **closed formula**, also known as a **sentence**, is a formula ϕ such that $\text{free}(\phi) = \emptyset$. An n**-formula** ϕ is a formula such that $\text{free}(\phi)$ has n elements. Thus, a closed formula has no free variables while an n-formula has n free variables.

2.8 The Satisfaction Relation, Assignments, and Interpretations

An **assignment** in an S-structure $\mathcal{A}$ is a map $\mathfrak{a} : \{v_n : n \in \mathbb{N}\} \to A$ of the set of variables into the universe of $\mathcal{A}$.

An S-**interpretation** $\mathfrak{I}$ is a pair $(\mathcal{A}, \mathfrak{a})$ consisting of an S-structure $\mathcal{A}$ and an assignment $\mathfrak{a}$ in $\mathcal{A}$.

If $\mathfrak{a}$ is an assignment in $\mathcal{A}$, $a \in A$, and x is a variable, then let $\mathfrak{a}\frac{a}{x}$ be the assignment in $\mathcal{A}$ defined as follows:

$$\mathfrak{a}\frac{a}{x}(y) := \begin{cases} \mathfrak{a}(y) & \text{if } y \neq x, \\ a & \text{if } y = x. \end{cases}$$

If $\mathfrak{I} = (\mathcal{A}, \mathfrak{a})$ let $\mathfrak{I}\frac{a}{x} := \left(\mathcal{A}, \mathfrak{a}\frac{a}{x}\right)$.

Definition Fix a symbol set S. For a variable x, let $\mathfrak{I}(x) := \mathfrak{a}(x)$. For a constant symbol $c \in S$, let $\mathfrak{I}(c) := c^{\mathcal{A}}$. For a k-ary function symbol $f \in S$ and terms τ_1,..., τ_k, let

$$\mathfrak{I}(f\tau_1 \ldots \tau_k) := f^{\mathcal{A}}(\mathfrak{I}(\tau_1) \ldots \mathfrak{I}(\tau_k)).$$

This definition effectively defines $\mathfrak{I}(\tau)$ for a term τ.

Definition Let τ_1,...,τ_k be terms, ϕ, ϕ_1, and ϕ_2 be formulas, and R a k-ary relation symbol all with respect to a symbol set S. We say that $\mathfrak{I} = (\mathcal{A}, \mathfrak{a})$ is a **model** of ϕ, $\mathfrak{I}$ **satisfies** ϕ, or that ϕ **holds** in $\mathfrak{I}$, and write $\mathfrak{I} \models \phi$, if

$\mathfrak{I} \models \tau_1 = \tau_2$	iff	$\mathfrak{I}(\tau_1) = \mathfrak{I}(\tau_2)$
$\mathfrak{I} \models R\tau_1 \ldots \tau_k$	iff	$R^{\mathcal{A}}\mathfrak{I}(\tau_1) \ldots \mathfrak{I}(\tau_k)$
$\mathfrak{I} \models \neg\phi$	iff	not $\mathfrak{I} \models \phi$
$\mathfrak{I} \models (\phi_1 \wedge \phi_2)$	iff	$\mathfrak{I} \models \phi_1$ and $\mathfrak{I} \models \phi_2$
$\mathfrak{I} \models (\phi_1 \vee \phi_2)$	iff	$\mathfrak{I} \models \phi_1$ or $\mathfrak{I} \models \phi_2$
$\mathfrak{I} \models (\phi_1 \to \phi_2)$	iff	if $\mathfrak{I} \models \phi_1$ then $\mathfrak{I} \models \phi_2$
$\mathfrak{I} \models (\phi_1 \leftrightarrow \phi_2)$	iff	$\mathfrak{I} \models \phi_1$ if and only if $\mathfrak{I} \models \phi_2$
$\mathfrak{I} \models \forall x\phi$	iff	for all $a \in A$, $\mathfrak{I}\frac{a}{x} \models \phi$
$\mathfrak{I} \models \exists x\phi$	iff	there is an $a \in A$ such that $\mathfrak{I}\frac{a}{x} \models \phi$

2.9 Structures as Models, more on Satisfaction

The following fact is a corollary to what is known as the "Coincidence Lemma," which can be found in [2 pg. 35]: for an S-formula ϕ and an S-interpretation $\mathfrak{I} = (\mathcal{A}, \mathfrak{a})$, the validity of ϕ under $\mathfrak{I}$ depends only on the assignments for the finitely many variables occurring free in ϕ and on the interpretation of the symbols of S in $\mathcal{A}$. If these variables are among v_1,...,v_n, it is at most the $\mathfrak{a}$-values $a_i = \mathfrak{a}(v_i)$ for $i = 1, \ldots, n$ which are significant. Thus, instead of $(\mathcal{A}, \mathfrak{a}) \models \phi$, we shall often use the more suggestive notation

$$\mathcal{A} \models \phi(a_1, \ldots, a_n).$$

When ϕ is a sentence, we will write

$$\mathcal{A} \models \phi,$$

without even mentioning an assignment. In that case we say that $\mathcal{A}$ is a **model** of ϕ. For a set of sentences Φ, $\mathcal{A} \models \Phi$ means that $\mathcal{A} \models \phi$ for every $\phi \in \Phi$.

$\mathcal{A} \models \phi(a_1, \ldots, a_n)$ denotes that structure $\mathcal{A}$ **satisfies** formula ϕ (i.e., ϕ is true in $\mathcal{A}$) when the variables $\text{free}(\phi) = \{v_1, \ldots, v_n\}$ are replaced by the values $a_1, \ldots, a_n$ everywhere in ϕ.

2.10 Symbolic Manipulation in Formulas

Suppose that S_1 and S_2 are two symbol sets. A map $\sigma : S_1 \to S_2$ is called **structure preserving** if the following conditions hold:

1. if $c \in S_1$ is a constant symbol then $\sigma(c)$ is a constant symbol in S_2.

2. if f is a $k-$ary function symbol in S_1 then $\sigma(f)$ is a $k-$ary function symbol in S_2.

3. if R is a $k-$ary relation symbol in S_1 then $\sigma(R)$ is a $k-$ary relation symbol in S_2.

Now suppose σ is a structure preserving map from S_1 to S_2. Let ϕ be an S_1-formula.

Definition The S_2-formula $\sigma(\phi)$ to be the result of making the following replacements:

- Every constant symbol $c \in S_1$ that occurs in ϕ is replaced by $\sigma(c)$.

- Every $k-$ary function symbol $f \in S_1$ that occurs in ϕ is replaced by $\sigma(f)$.

- Every $k-$ary relation symbol $R \in S_1$ that occurs in ϕ is replaced by $\sigma(R)$.

2.11 Generalization of Isomorphism Concept

We present a definition of isomorphism between structures that don't necessarily have the same symbol set.

Let $\mathcal{A}$ be an S_1-structure and $\mathcal{B}$ be an S_2-structure. We say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if there are two maps, $\sigma : S_1 \to S_2$ and $g : A \to B$ such that all of the following conditions hold:

1. σ is a bijection, i.e., S_1 and S_2 are equipollent via the function σ.

2. σ is structure preserving.

3. g is a bijection such that:

   (a) for all constant symbols $c \in S_1$, $g\left(c^{\mathcal{A}}\right) = \sigma(c)^{\mathcal{B}}$.

(b) if $f \in S_1$ is a k-ary function symbol and $(a_1, ..., a_k)$ is any tuple in A^k then

$$g\left(f^{\mathcal{A}}(a_1, ..., a_k)\right) = \sigma(f)^{\mathcal{B}}(g(a_1), ..., g(a_k)).$$

(c) if $R \in S_1$ is a k-ary relation symbol and $(a_1, ..., a_k)$ is any tuple in A^k then

$$R^{\mathcal{A}}(a_1, ..., a_k) \Longleftrightarrow \sigma(R)^{\mathcal{B}}(g(a_1), ..., g(a_k)).$$

Theorem This definition of isomorphism indeed generalizes the concept of isomorphism when the two structures have the same symbol set. In other words, if $\mathcal{A}$ and $\mathcal{B}$ are $S-$structures, then they are isomorphic in this sense if and only if they are isomorphic in the usual sense.

Proof. Note that we can take σ to be the identity function on S. In that event, $\sigma(c)^{\mathcal{B}} = c^{\mathcal{B}}$, $\sigma(f)^{\mathcal{B}} = f^{\mathcal{B}}$, and $\sigma(R)^{\mathcal{B}} = R^{\mathcal{B}}$.

2.12 Elementary Embedding Generalization

Let $\mathcal{M}$ be an S_1-structure and $\mathcal{N}$ be an S_2-structure. A function $F : M \to N$ is called an **elementary embedding** if and only if there is a structure preserving $\sigma : S_1 \to S_2$ such that for all $n - S_1-$formulas ϕ and for all $m_1, ..., m_n$ in M,

$$\mathcal{M} \models \phi(m_1, ..., m_n) \Longleftrightarrow \mathcal{N} \models \sigma(\phi)(F(m_1), ..., F(m_n)).$$

If $\mathcal{M}$ and $\mathcal{N}$ are two structures, we say $\mathcal{M}$ can be **elementarily embedded** in $\mathcal{N}$ if there is such an elementary embedding. Denote this by $\mathcal{M} \prec \mathcal{N}$.

Theorem This definition of elementary embedding indeed generalizes the concept of elementary embedding when the two structures have the same symbol set. Note that we can again take σ to be the identity function on the underlying symbol set.

3 Reduced Products of Structures

Now we proceed to a variant of a reduced product of structures. Much of this material is drawn from [3]. The path toward a structure with the property that all structures, regardless of their symbol set, are elementarily embeddable within it, is to provide a suitable universe paired with a suitable structure map.

Fix a family of symbol sets $\{S_i : i \in I\}$ where I is a nonempty set and that for all $i \in I$, $\mathcal{A}_i$ is an S_i-structure. Define $\mathbb{S} = \bigcup_{i \in I} S_i$. Then $\mathcal{A}$, the reduced product of these structures, is defined as follows:

1. (Definition of the Universe) The universe A of $\mathcal{A}$ is the set functions defined as follows. Let $\prod_{i \in I} A_i$ denote the set

$$\prod_{i \in I} A_i := \left\{ \overrightarrow{a} \in \left[I \to \bigcup_{i \in I} A_i \right] : \forall i (\vec{a}(i) \in A_i) \right\}.$$

A, the universe of $\mathcal{A}$, is defined to be $\prod_{i\in I} A_i$. Recall that $[x \to y]$ is the set of all functions from x to y. This is the usual **Cartesian product** of the family $\{A_i : i \in I\}$. Elements of the universe of $\mathcal{A}$ are functions whose domain is I and the i^{th} coordinate is an element of A_i.

2. (Structure mapping of Constant Symbols) For every constant symbol c in $\mathbb{S}$, let $\overrightarrow{c}$ be the function that maps each $i \in I$ to $c^{\mathcal{A}_i}$, and let $c^{\mathcal{A}} = \overrightarrow{c}$.

3. (Structure mapping of Function Symbols) If f is a k-ary function symbol in $\mathbb{S}$, define $f^{\mathcal{A}}$ as follows. For k elements $\overrightarrow{a}_1, ..., \overrightarrow{a}_k$ in A, let $f^{\mathcal{A}}(\overrightarrow{a}_1, ..., \overrightarrow{a}_k)$ be the function that maps each $i \in I$ to $f^{\mathcal{A}_i}(\overrightarrow{a}_1(i), ..., \overrightarrow{a}_k(i))$, i.e., $f^{\mathcal{A}}(\overrightarrow{a}_1, ..., \overrightarrow{a}_k)(i) := f^{\mathcal{A}_i}(\overrightarrow{a}_1(i), ..., \overrightarrow{a}_k(i))$.

4. (Structure mapping of Relation Symbols) If R is a k-ary relation symbol in $\mathbb{S}$, then $R^{\mathcal{A}}$ is defined to be the following set
$$R^{\mathcal{A}} := \left\{(\overrightarrow{a}_1, ..., \overrightarrow{a}_k) \in A^k : \forall i\left((\overrightarrow{a}_1(i), ..., \overrightarrow{a}_k(i)) \in R^{\mathcal{A}_i}\right)\right\}.$$
In other words, $(\overrightarrow{a}_1, ..., \overrightarrow{a}_k) \in R^{\mathcal{A}}$ if and only if for all indices $i \in I$,
$$(\overrightarrow{a}_1(i), ..., \overrightarrow{a}_k(i)) \in R^{\mathcal{A}_i}.$$

We now have all the ingredients for the trivial reduced product: a universe and a structure map whose domain is a symbol set consisting of constant symbols, function symbols, and relation symbols. Denote this structure by $\prod_{i\in I} \mathcal{A}_i =: \mathcal{A}$. $\mathcal{A}$ is an $\mathbb{S}-$structure.

The following theorem will serve as an important step in proving the main result.

Theorem Fix a family of symbol sets $\{S_i : i \in I\}$ where I is a nonempty set and that for all $i \in I$, $\mathcal{A}_i$ is an S_i-structure. Let $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$. Then for all $i \in I$, $\mathcal{A}_i$ can be elementarily embedded in $\mathcal{A}$ via an injection.

Proof. Fix an i and without loss of generality, assume ϕ is an arbitrary 1-formula with symbols occurring in S_i. Let σ be the identity function on S_i. Let $F_i : A_i \to A$ be a function defined as follows. Select an $\overrightarrow{a}$, an arbitrary element of A, and let $b \in A_i$. Define $F_i(b) \in A$ so that
$$F_i(b)(j) := \begin{cases} b & \text{if } j = i, \\ \overrightarrow{a}(j) & \text{if } j \neq i. \end{cases}$$
(There are as many ways of defining F_i as there are elements of A.) We will show that
$$\mathcal{A}_i \models \phi(b) \iff \mathcal{A} \models \sigma(\phi)(F_i(b)).$$

In other words, we will show that
$$\mathcal{A}_i \models \phi(b) \iff \mathcal{A} \models \phi(F_i(b)).$$

We will proceed by first proving the following lemma.

Lemma

$$\mathcal{A} \models \phi\left(F_i\left(b\right)\right) \Longleftrightarrow \forall j \in I\left(\mathcal{A}_j \models \phi\left(F_i\left(b\right)\left(j\right)\right)\right).$$

Proof. We proceed by induction. Step 1: ϕ is atomic.

- ϕ is $\tau_1 = \tau_2$. Let $\mathfrak{I} = (\mathcal{A}, \mathfrak{a})$ be an $\mathbb{S}-$interpretation associated with the structure $\mathcal{A}$. Note that $\mathcal{A} \models \phi\left(F_i\left(b\right)\right)$ if and only if

$$\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(\tau_1\right) = \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(\tau_2\right).$$

  Suppose τ_1 is $f_1 t_1 ... t_p$ and τ_2 is $f_2 u_1 ... u_q$, where each t and u is an $\mathbb{S}-$term and that f_1 and f_2 are $p-$ary and $q-$ary function symbols, respectively. Furthermore, allow p and q to be zero in which case interpret τ_1 and/or τ_2 as constants. Then $\mathfrak{I}\frac{F_i(b)}{x}\left(\tau_1\right) = \mathfrak{I}\frac{F_i(b)}{x}\left(\tau_2\right)$ if and only if

$$f_1^{\mathcal{A}}\left(\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(t_1\right), ..., \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(t_p\right)\right) = f_2^{\mathcal{A}}\left(\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(u_1\right), ..., \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(u_q\right)\right).$$

By the definition of the reduced product, for all $j \in I$, the preceding equation holds if and only if

$$f_1^{\mathcal{A}_j}\left(\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(t_1\right), ..., \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(t_p\right)\right)\left(j\right) = f_2^{\mathcal{A}_j}\left(\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(u_1\right), ..., \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(u_q\right)\right)\left(j\right).$$

Thus,

$$\mathcal{A} \models \phi\left(F_i\left(b\right)\right) \Longleftrightarrow \forall j \in I\left(\mathcal{A}_j \models \phi\left(F_i\left(b\right)\left(j\right)\right)\right).$$

- $\phi = R\tau_1 ... \tau_k$. Note that $\mathfrak{I} \models \phi\left(F_i\left(b\right)\right)$ if and only if

$$R^{\mathcal{A}}\mathfrak{I}\frac{F_i\left(b\right)}{x}\left(\tau_1\right) ... \mathfrak{I}\frac{F_i\left(b\right)}{x}\left(\tau_k\right).$$

  By the definition of the reduced product, the preceding expression holds if and only if for all $j \in I$,

$$R^{\mathcal{A}_j}\mathfrak{I}\frac{F_i\left(b\right)\left(j\right)}{x}\left(\tau_1\right) ... \mathfrak{I}\frac{F_i\left(b\right)\left(j\right)}{x}\left(\tau_k\right).$$

Thus,

$$\mathcal{A} \models \phi\left(F_i\left(b\right)\right) \Longleftrightarrow \forall j \in I\left(\mathcal{A}_j \models \phi\left(F_i\left(b\right)\left(j\right)\right)\right).$$

This completes the proof for atomic formulas.
Step 2: induction.

- $\phi = \neg\psi$. Suppose $\mathcal{A} \models \neg\psi\left(F_i\left(b\right)\right)$ and let $j \in I$ be arbitrary. Since it is not the case that $\mathcal{A} \models \psi\left(F_i\left(b\right)\right)$, it is not the case that $\mathcal{A}_j \models \psi\left(F_i\left(b\right)\left(j\right)\right)$ by induction. Consequently, $\mathcal{A}_j \models \neg\psi\left(F_i\left(b\right)\left(j\right)\right)$. As j was arbitrary, $\forall j \in I\left(\mathcal{A}_j \models \neg\psi\left(F_i\left(b\right)\left(j\right)\right)\right)$. For the converse, assume that $\forall j \in I\left(\mathcal{A}_j \models \neg\psi\left(F_i\left(b\right)\left(j\right)\right)\right)$. To arrive at a contradiction, suppose it is not the case that $\mathcal{A} \models \neg\psi\left(F_i\left(b\right)\right)$. Then $\mathcal{A} \models \psi\left(F_i\left(b\right)\right)$. By induction, $\forall j \in I\left(\mathcal{A}_j \models \psi\left(F_i\left(b\right)\left(j\right)\right)\right)$ which contradicts our assumption.

- $\phi = \phi_1 \vee \phi_2$. Suppose $\mathcal{A} \models (\phi_1 \vee \phi_2)(F_i(b))$ and let $j \in I$ be arbitrary. Either $\mathcal{A} \models \phi_1(F_i(b))$ or $\mathcal{A} \models \phi_2(F_i(b))$, or both. Without loss of generality, assume $\mathcal{A} \models \phi_1(F_i(b))$. By induction, $\mathcal{A}_j \models \phi_1(F_i(b)(j))$. Consequently, $\mathcal{A}_j \models (\phi_1 \vee \phi_2)(F_i(b)(j))$ follows. Again, since j was arbitrary, we have $\forall j \in I(\mathcal{A}_j \models (\phi_1 \vee \phi_2)(F_i(b)(j)))$. For the converse, assume that $\forall j \in I(\mathcal{A}_j \models (\phi_1 \vee \phi_2)(F_i(b)(j)))$. To arrive at a contradiction, suppose that it is not the case that $\mathcal{A} \models (\phi_1 \vee \phi_2)(F_i(b))$. From that, we conclude that $\mathcal{A} \models (\neg\phi_1 \wedge \neg\phi_2)(F_i(b))$. Consequently, $\mathcal{A} \models \neg\phi_1(F_i(b))$ and $\mathcal{A} \models \neg\phi_2(F_i(b))$. We have shown above that $\mathcal{A} \models \neg\phi_1(F_i(b))$ entails $\forall j \in I(\mathcal{A}_j \models \neg\phi_1(F_i(b)(j)))$ and in addition that $\mathcal{A} \models \neg\phi_2(F_i(b))$ entails $\forall j \in I(\mathcal{A}_j \models \neg\phi_2(F_i(b)(j)))$ from which we can conclude that $\forall j \in I(\mathcal{A}_j \models (\neg\phi_1 \wedge \neg\phi_2)(F_i(b)(j)))$ which contradicts our assumption.

- $\phi = \exists x\psi$. If $\mathcal{A} \models \exists x\psi(F_i(b), x)$, then there is a $\overrightarrow{c} \in A$ such that $\mathcal{A} \models \psi(F_i(b), \overrightarrow{c})$. By induction, $\forall j \in I(\mathcal{A}_j \models \psi(F_i(b)(j), \overrightarrow{c}(j)))$. Consequently, $\forall j \in I(\mathcal{A}_j \models \exists x\psi(F_i(b)(j), x))$. For the converse, assume that $\forall j \in I(\mathcal{A}_j \models \exists x\psi(F_i(b)(j), x))$. To arrive at a contradiction, assume it is not the case that $\mathcal{A} \models \exists x\psi(F_i(b), x)$. Then $\mathcal{A} \models \forall x\neg\psi(F_i(b), x)$. Thus for every $\overrightarrow{c} \in A$ we have $\mathcal{A} \models \neg\psi(F_i(b), \overrightarrow{c})$. As we have seen in the first part of the induction step, this implies that for every $\overrightarrow{c} \in A$ we have $\forall j \in I(\mathcal{A}_j \models \neg\psi(F_i(b)(j), \overrightarrow{c}(j)))$; so $\forall j \in I(\mathcal{A}_j \models \forall x\neg\psi(F_i(b)(j), x))$, contradicting our assumption.

This completes the proof of the lemma.

Returning to the proof of the theorem, suppose that $\mathcal{A}_i \models \phi(b)$ and, to arrive at a contradiction, that $\mathcal{A} \not\models \phi(F_i(b))$. $\mathcal{A} \not\models \phi(F_i(b))$ implies $\mathcal{A} \models \neg\phi(F_i(b))$. By the lemma,

$$\forall j \in I(\mathcal{A}_j \models \neg\phi(F_i(b)(j))).$$

In particular, $\mathcal{A}_i \models \neg\phi(F_i(b)(i))$, implying $\mathcal{A}_i \models \neg\phi(b)$ which contradicts the assumption $\mathcal{A}_i \models \phi(b)$.
For the other direction, suppose $\mathcal{A} \models \phi(F_i(b))$. By the lemma, $\mathcal{A}_i \models \phi(F_i(b)(i))$. Consequently, $\mathcal{A}_i \models \phi(b)$.
To show that F_i is injective, let b and b' be elements of A_i and suppose $F_i(b) = F_i(b')$. That entails $\forall j \in I(F_i(b)(j) = F_i(b')(j))$. In particular, $F_i(b)(i) = F_i(b')(i)$, implying $b = b'$. $\square$

Part III
Building an Ultimate Superstructure

We seek now to define a structure $\mathcal{U}$ with the property that all structures are elementarily embeddable within it. To do this we will first obtain the set of all structures which is made possible by the stratified comprehension theorem in part 2, section 1. We will form the reduced product of all structures. To that end we will stipulate some definitions and then proceed to form the reduced product of all structures.

Theorem The set I of all structures, inclusive of all symbol sets, exists.

Proof. Given an S-structure $\mathcal{A} = (A, \alpha)$, the formula $\Sigma(S, A, \alpha)$, to be defined shortly, will be satisfied if and only if α is an S-structure map from symbol set S to the union of A (constants), the set of all k-ary relations on A, and the set of all k-ary functions on A.

Define the following sets A' and A'':

Let A' be the union of all k-ary functions on A for $k \geq 1$, i.e.,

$$A' := \bigcup_{k \geq 1} \left[A^k \to A\right].$$

Let A'' be the union of all k-ary relations on A for $k \geq 1$, i.e.,

$$A'' := \bigcup_{k \geq 1} \mathcal{P}\left(A^k\right).$$

Observe that for a symbol set S, α is an S-structure map if and only if the *conjunction* of the following sub-formulas is true:

$$\alpha \in [S \to (A \cup A' \cup A'')]$$

$$(\alpha \upharpoonright S_c) \in [S_c \to A]$$

$$(\alpha \upharpoonright S_f) \in [S_f \to A']$$

$$(\alpha \upharpoonright S_R) \in [S_R \to A''],$$

where S_c, S_f, and S_R are the possibly empty set of constant symbols, the possibly empty set of function symbols, and the possibly empty set of relation symbols in S, respectively. $\upharpoonright$ denotes function restriction.

Let $\Sigma(S, A, \alpha)$ be the aforementioned formula (the conjunction of those four formulas). $\Sigma(S, A, \alpha)$ is satisfied if and only if α is an S-structure map on an S-structure whose universe is A.

Define $\phi(S, x)$ to be the formula $\exists A \exists \alpha\, (x = (A, \alpha) \wedge \Sigma(S, A, \alpha))$. $\phi(S, x)$ is satisfied if and only if x is an S-structure.

Finally, let $I = \{x : \exists S\, (\phi(S, x))\}$. Thus, the set of all structures I exists by the stratified comprehension theorem mentioned in part 2, section 1.$\square$

Definition To form the reduced product of all structures, define, for each $i \in I$, $\mathcal{A}_i := i$. Let $\mathcal{U} = \prod_{i \in I} \mathcal{A}_i$.

The following theorem is the primary result of this paper.

Theorem Every structure, regardless of symbol set, can be elementarily embedded within $\mathcal{U}$ via an injection.

Proof. This follows from the theorem in section 3 of Part II.□

Part IV
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