2

When one multiplies (21.83) by $dt\, dx^1\, dx^2\, dx^3$ and integrates to obtain the action integral, the term (21.85) integrates out to a surface term. Variations of the geometry interior to this surface make no difference in the value of this surface term. Therefore it has no influence on the equations of motion to drop the term (21.85). The result of the calculation (exercise 21.10) is simple: what is left over after dropping the divergence merely changes the sign of the terms in $\mathrm{Tr}\,\boldsymbol{K}^2$ and $(\mathrm{Tr}\,\boldsymbol{K})^2$ in (21.84). Thus the variation principle becomes

$$\begin{aligned}(\text{extremum}) &= I_{\text{modified}} = \int \mathcal{L}_{\text{modified}}\, d^4x \\ &= (1/16\pi)\int [R + (\boldsymbol{n}\cdot\boldsymbol{n})((\mathrm{Tr}\,\boldsymbol{K})^2 - \mathrm{Tr}\,\boldsymbol{K}^2)]Ng^{1/2}\, dt\, d^3x + \int \mathcal{L}_{\text{fields}}\, d^4x.\end{aligned} \tag{21.86}$$

This expression, rephrased, is the starting point for Arnowitt, Deser, and Misner's analysis of the dynamics of geometry.

Two supplements from a paper of York (1972b; see also exercise 21.9) enlarge one's geometric insight into what is going on in the foregoing analysis. First, the tensor of extrinsic curvature lets itself be defined [see also Fischer (1971)] most naturally in the form

$$\boldsymbol{K} = -\frac{1}{2}\boldsymbol{\pounds}_{\boldsymbol{n}}\boldsymbol{g}, \tag{21.87}$$

where $\boldsymbol{g}$ is the metric tensor of the 3-geometry, $\boldsymbol{n}$ is the timelike unit normal field, and $\boldsymbol{\pounds}$ is the Lie derivative as defined in exercise 21.8. Second, the divergence (21.85), which has to be added to the Lagrangian of (21.86) to obtain the full Hilbert Lagrangian, is

$$-2[(-{}^{(4)}g)^{1/2}(n^{\alpha'}\mathrm{Tr}\,\boldsymbol{K} + a^{\alpha'})]_{,\alpha'}, \tag{21.88}$$

where the coordinates are general (see exercise 21.10), and

$$a^{\alpha'} = n^{\alpha'}{}_{;\beta'}n^{\beta'} \tag{21.89}$$

is the 4-acceleration of an observer traveling along the timelike normal $\boldsymbol{n}$ to the successive slices.

§21.7. THE ARNOWITT, DESER, AND MISNER FORMULATION OF THE DYNAMICS OF GEOMETRY

Dirac (1959, 1964, and earlier references cited therein) formulated the dynamics of geometry in a (3 + 1)-dimensional form, using generalizations of Poisson brackets and of Hamilton equations. Arnowitt, Deser, and Misner instead made the Hilbert-Palatini variational principle the foundation for this dynamics. Because of its simplicity, this ADM (1962) approach is followed here. The gravitational part of the integrand in the Hilbert-Palatini action principle is rewritten in the condensed but standard form (after inserting a 16π that ADM avoid by other units) as

$$\begin{aligned}16\pi\mathcal{L}_{\text{geom true}} = \mathcal{L}_{\text{geom ADM}} &= -g_{ij}\,\partial\pi^{ij}/\partial t - N\mathcal{H} - N_i\mathcal{H}^i \\ &\quad - 2\left[\pi^{ij}N_j - \frac{1}{2}N^i\,\mathrm{Tr}\,\boldsymbol{\pi} + N^{|i}(g)^{1/2}\right]_{,i}.\end{aligned} \tag{21.90}$$

Here each item of abbreviation has its special meaning and will play its special part, a part foreshadowed by the name now given it:

$$\pi^{ij}_{\text{true}} = \frac{\delta(\text{action})}{\delta g_{ij}} = \begin{pmatrix} \text{“geometrodynamic} \\ \text{field momentum” dyn-} \\ \text{amically conjugate to} \\ \text{the “geometrodynamic} \\ \text{field coordinate” } g_{ij} \end{pmatrix} = \frac{\pi^{ij}}{16\pi};\ \pi^{ij} = g^{1/2}(g^{ij}\operatorname{Tr}\boldsymbol{K} - K^{ij}) \tag{21.91}$$

Momenta conjugate to the dynamic g_{ij}

(here the π^{ij} of ADM is usually more convenient than π^{ij}_{true}); and

$$\begin{aligned} \mathcal{H}_{\text{true}} &= \mathcal{H}(\pi^{ij}_{\text{true}}, g_{ij}) = (\text{“super-Hamiltonian”}) = \mathcal{H}/16\pi; \\ \mathcal{H}(\pi^{ij}, g_{ij}) &= g^{-1/2}\left(\operatorname{Tr}\boldsymbol{\pi}^2 - \frac{1}{2}(\operatorname{Tr}\boldsymbol{\pi})^2\right) - g^{1/2}R; \end{aligned} \tag{21.92}$$

and

$$16\pi\mathcal{H}^i_{\text{true}} = \mathcal{H}^i = \mathcal{H}^i(\pi^{ij}, g_{ij}) = (\text{“supermomentum”}) = -2\pi^{ik}{}_{|k}. \tag{21.93}$$

Here the covariant derivative is formed treating π^{ik} as a tensor density, as its definition in (21.91) shows it to be (see §21.2). The quantities to be varied to extremize the action are the coefficients in the metric of the 4-geometry, as follows: the six g_{ij} and the lapse function N and shift function N_i; and also the six "geometrodynamic momenta," π^{ij}. To vary these momenta as well as the metric is (1) to follow the pattern of elementary Hamiltonian dynamics (Box 21.1), where, by taking the momentum p to be as independently variable as the coordinate x, one arrives at two Hamilton equations of the first order instead of one Lagrange equation of the second order, and (2) to follow in some measure the lead of the Palatini variation principle of §21.2. There, however, one had 40 connection coefficients to vary, whereas here one has come down to only six π^{ij}. To know these momenta and the 3-metric is to know the extrinsic curvature. Before carrying out the variation, drop the divergence $-2[\quad]_{,i}$ from (21.90), since it gives rise only to surface integrals and therefore in no way affects the equations of motion that will come out of the variational principle. Also rewrite the first term in (21.90) in the form

$$-(\partial/\partial t)(g_{ij}\pi^{ij}) + \pi^{ij}\,\partial g_{ij}/\partial t, \tag{21.94}$$

and drop the complete time-derivative from the variation principle, again because it is irrelevant to the resulting equations of motion. The action principle now takes the form

$$\begin{aligned} \text{extremum} &= I_{\text{true}} = I_{\text{ADM}}/16\pi \\ &= (1/16\pi)\int [\pi^{ij}\,\partial g_{ij}/\partial t - N\mathcal{H}(\pi^{ij}, g_{ij}) - N_i\mathcal{H}^i(\pi^{ij}, g_{ij})]\, d^4x \\ &\qquad + \int \mathcal{L}_{\text{field}}\, d^4x. \end{aligned} \tag{21.95}$$

The action principle itself, here as always, tells one what must be fixed to make the action take on a well-defined value (if and when the action possesses an extremum). Apart from appropriate potentials having to do with fields other than geom-