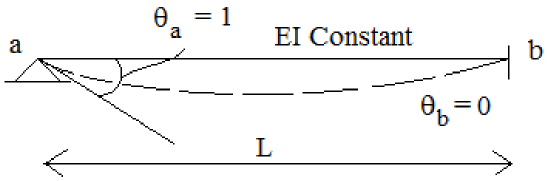
Moment distribution method – Joint moments in a frame



Take the propped-cantilevered beam shown. It has no load. We'd like to know the general relationship between θ_a and M_{ab} (end moment of member ab at "a") or M_{ba} (end moment of member ab at "b")

$$EI\frac{d^4\upsilon}{dx^4} = \omega = 0$$

Initial conditions: $\upsilon(0) = 0$ $\upsilon(L) = 0$ $\upsilon'(0) = \theta_a$ $\upsilon'(L) = 0$

$\upsilon = x(1-\frac{x}{L})^2\theta_a$ (skipped work)

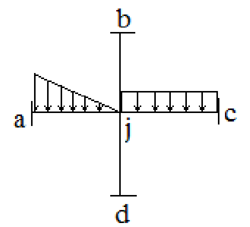
$\frac{d^2\upsilon}{dx^2} = \frac{-M}{EI} \Rightarrow M(x) = E\frac{I}{L}(4-6\frac{x}{L})\theta_a$ (this is true of any load)

So, $M_{ab} = 4E\frac{I}{L}\theta_a$ and $M_{ba} = 2E\frac{I}{L}\theta_a$

OR

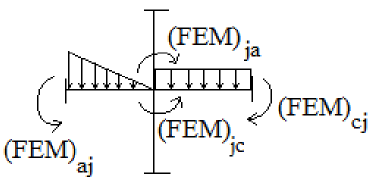
$M_{ab} = S_{ab}\theta_a$, where $S_{ab} = S_{ba} =$ member stiffness $= 4Ek_{ab}$

where $k_{ab} = k_{ba} =$ stiffness factor $= \frac{I}{L}$



(a)

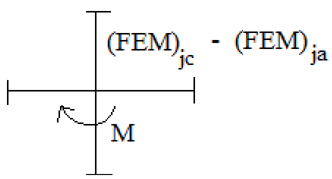
Lock the structure so that there are four fixed-end beams. Find the FEMs, including the total moment M at the center (ccw). Create an opposite moment shown (cw moment M) to “unlock” the beam. Joint j now rotates through an angle θ. Now, picture (e) is equivalent to picture (a) and we can proceed with the moment distribution.



$$\begin{aligned} m_{ja} &= S_{ja}\theta \\ m_{jb} &= S_{jb}\theta \\ m_{jc} &= S_{jc}\theta \\ m_{jd} &= S_{jd}\theta \end{aligned} \qquad (1)$$

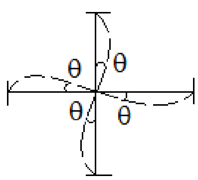
Sum of the moments must equal zero, so, $(S_{ja} + S_{jb} + S_{jc} + S_{jd})\,\theta = M$, where M is the “external” moment at joint j

(all θ are equal due to continuity)



If the members have the same E, but not necessarily the same I or L, then $4E(k_{ja} + k_{jb} + k_{jc} + k_{jd})\,\theta = M \Rightarrow \theta = \dfrac{M}{4E\sum k}$ (2)

$\sum k$ includes all members that connect at rotating joint (can vary depending on which end of beam)



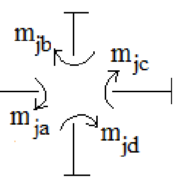
From (1) and (2),

$$m_{ja} = S_{ja}\frac{M}{4E\sum k} = \frac{4EI}{L}\frac{M}{4E\sum k} = \frac{k_{ja}}{\sum k}M = D_{ja}M$$

$$m_{jb} = \frac{k_{jb}}{\sum k}M = D_{jb}M$$

$$m_{jc} = \frac{k_{jc}}{\sum k}M = D_{jc}M$$

$$m_{jd} = \frac{k_{jd}}{\sum k}M = D_{jd}M$$



(e)

These are called distributed moments (DMs). D = distribution factor = $\dfrac{k}{\sum k}$

Assumes constant E – usually the case since beams made of different materials are rarely connected together.

note: D depends only on member dimensions. The individual moments are just ratios of each other that add up to M – i.e. the “external” moment, M, is *distributed* among the connecting beams, according to their relative dimensions (stiffnesses).

$m_{aj}, m_{bj}, m_{cj}, m_{dj}$ - called the carry-over-moments, need to be found.

M_{ab} and M_{ba}, which were found on the previous page, can be equated with a “carry-over-factor”; $M_{ba} = C_{ab}M_{ab}$; $C_{ab} = C_{ba} = \dfrac{1}{2}$

So, $\quad m_{aj} = \frac{1}{2} m_{ja}\ ;\ m_{bj} = \frac{1}{2} m_{jb}\ ;\ m_{cj} = \frac{1}{2} m_{jc}\ ;\ m_{dj} = \frac{1}{2} m_{jd}$

These are called carry-over-moments (COMs).

$$M_{ja} = (FEM)_{ja} + m_{ja}\ (= DM) + m_{ja}\ (= COM)$$

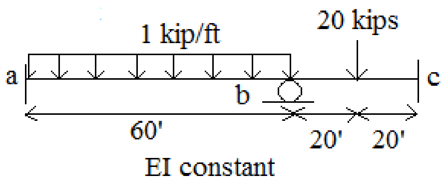
$$M_{aj} = (FEM)_{aj} + m_{aj}\ (= DM) + m_{aj}\ (= COM)$$

note: It is not yet clear how to find end moments for a frame when there is more than one joint that can rotate

note: sign conventions will become clear in the following examples

e.g. 1

e.g. 1



$$\text{beam ba:}\quad \frac{I/60}{I\left(1/60 + 1/40\right)} = .4$$

$$\text{beam bc:}\quad \frac{I/40}{I\left(1/60 + 1/40\right)} = .6$$

Clockwise moments = positive (FEMs can be found in Appendix B)

$(FEM)_{ba} = 300 \quad (FEM)_{ab} = -300 \quad (FEM)_{bc} = -100 \quad (FEM)_{cb} = 100$ *(skipped work)*

$M_b = -(300 + (-100)) = -200$ *(Total FEM at b)*

Distributed moments: $m_{ba} = .4(-200) = -80 kips * ft\ ;\ m_{bc} = .6(-200) = -120 kips * ft$

Carry-over-moments: $m_{ab} = \frac{1}{2} m_{ba} = -40 kips * ft\ ;\ m_{cb} = \frac{1}{2} m_{bc} = -60 kips * ft$

(COM has same sign as DM)

	ab	ba	bc	cb
$\frac{k}{\Sigma k}$	0	.4	.6	0
FEM	-300	300	-100	100
DM		-80	-120	
COM	-40			-60
Σ	-340	220	-220	40

$M_{ab} = -340 kips * ft$

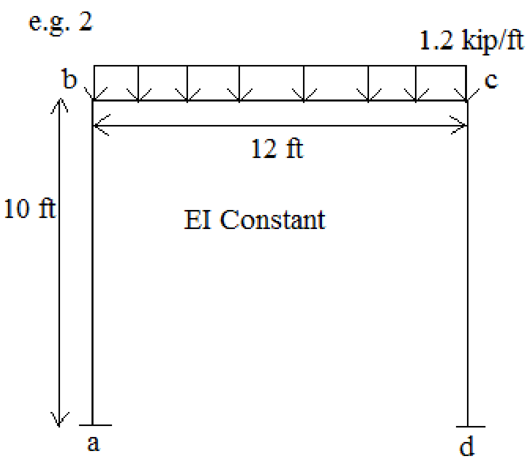
$M_{ba} = 220 kips * ft$

$M_{bc} = -220 kips * ft$

$M_{cb} = 40 kips * ft$

If there are multiple rotating joint, then the joints must be continually locked and unlocked until the carry-over-moments are considered negligible (see the following example) note: For all cycles : At a fixed support, DM is zero. At a joint across from a fixed support, COM is zero. For a span with no load, FEM is zero (this does not necessarily mean that M = 0 for that span).

e.g. 2



$$Beam\ ba = \frac{I/10}{I/10 + I/12} = .545 = Beam\ cd$$

$$Beam\ bc = \frac{I/12}{I/10 + I/12} = .4545 = Beam\ cb$$

$$(FEM)_{bc} = \frac{-1.2(12)^2}{12} = -14.4 kips * ft$$

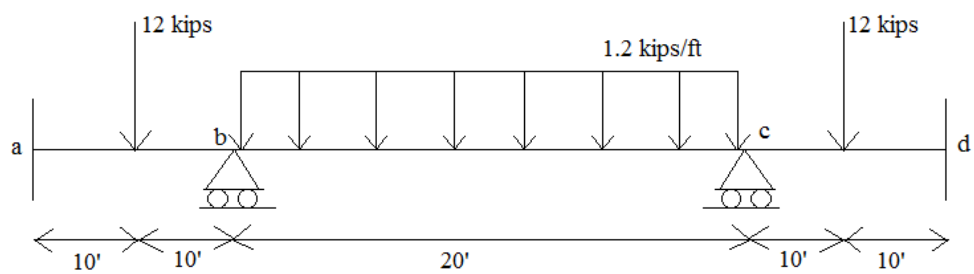
$$(FEM)_{cb} = 14.4 kips * ft$$

	(rigid)	(joint b)		(joint c)		(rigid)
	ab	ba	bc	cb	cd	dc
$k/\Sigma k$	0	.545	.4545	.4545	.545	0
FEM			-14.4	14.4		
DM		14.4(.545) = 7.85	14.4(.4545) = 6.545	-6.545	-7.85	
COM	7.85/2 = 3.93		-6.545/2 = -3.27	3.27		-3.93
DM		3.27(.545) = 1.785	3.27(.4545) = 1.49	-1.49	-1.785	
COM	1.785/2 = .8926		-1.49/2 = -.744	.744		-.8926
DM		.744(.545) = .406	.744(.4545) = .338	-.388	-.406	
COM	.406/2 = .203		-.338/2 = -.169	.169		-.203
DM		.169(.545) = .092	.169(.4545) = .077	-.077	-.092	
COM	.092/2 = .046		-.077/2 = -.038	.038		-.046
DM		.038(.545) = .021	.038(.4545) = .017	-.017	-.021	
COM	.021/2 = .01		-.017/2 = -.009	.009		-.01
DM		.009(.545) = .005	.009(.4545) = .004	-.004	-.005	
Σ	5.08	10.2	-10.2	10.2	-10.2	-5.08

(after 6 cycles) (kip * ft)

e.g. 3

e.g. 3



$\frac{k}{\Sigma k}$	ab	ba	bc	cb	cd	dc
	0	.5	.5	.5	.5	0
FEM	-30	30	-40	40	-30	30
DM		5	5			
COM	2.5			2.5		
DM				-6.25	-6.25	
COM			-3.13			-3.13
DM		1.57	1.57			
COM	.78			.78		
DM				-.39	-.39	
COM			-.20			-.20
DM		.10	.10			
COM	.05			.05		
DM				-.03	-.03	
Σ	-26.67	36.67	-36.67	36.67	-36.67	26.67

(kips * ft)

Since cb and cd have not yet been "balanced" by a distributed moment, the C.O.M. is added to M_c (10 + 2.5) = 12.5

From this point on, the C.O.M.'s become the sole FEMs, because all rotational joints have been balanced by distributed moments

OR
(release joints simultaneously)

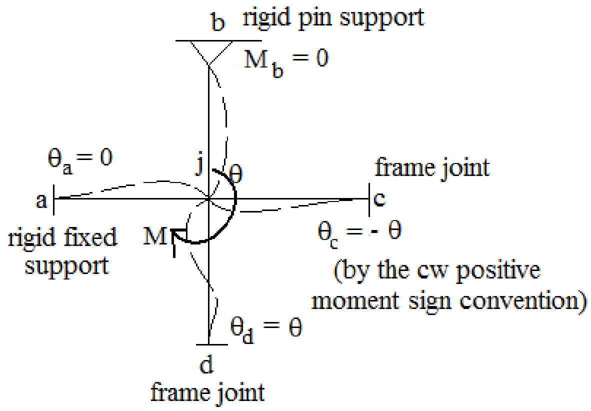
FEM	-30	30	-40	40	-30	30
DM		5	5	-5	-5	
COM	2.5		-2.5	2.5		-2.5
DM		1.25	1.25	-1.25	-1.25	
COM	.63		-.63	.63		-.63
DM		.32	.32	-.32	-.32	
COM	.16		-.16	.16		-.16
DM		.08	.08	-.08	-.08	
COM	.04		-.04	.04		-.04
DM		.02	.02	-.02	-.02	
Σ	-26.67	36.67	-36.67	36.67	-36.67	26.67

(kips * ft)

Joints are balanced. From this point on, the C.O.M.'s are the only FEMs.

note: So far, joint translations are ignored in the moment distribution method. This can have an effect on the accuracy of joint moments. Previous analysis of the one bay frame (with beam uniformly loaded) resulted in the exact solution because it is symmetrical (and hence there are no relative displacements in the columns), and there is no side sway, from inspection. Lateral loading and/or non-symmetrical gravity loading can cause joint translations.

Modified stiffness method – Shortcut for certain special cases



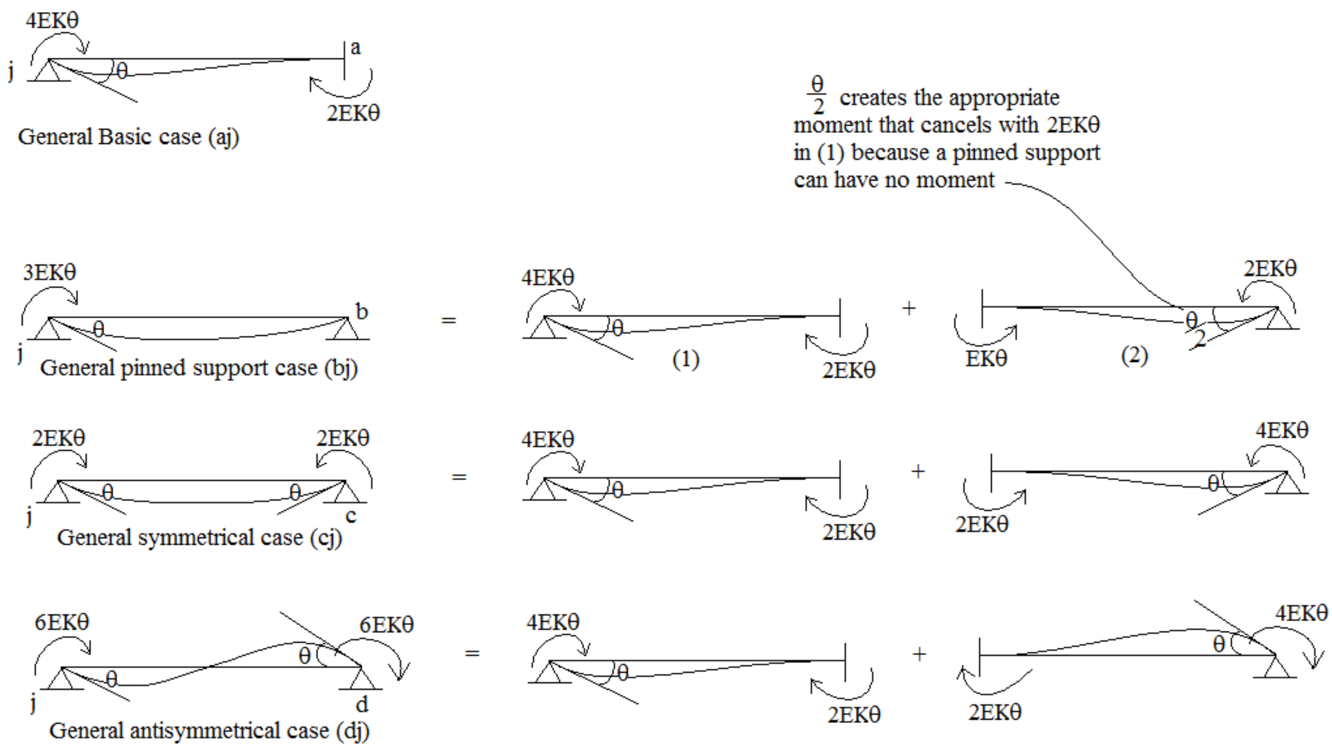
<u>Special cases</u> :

aj = the most basic member with a single DM at ja and COM at aj

bj = supported by a pin

cj = symmetrical

dj = antisymmetrical



Basic: $k_{ja}' = k_{ja}$, where k' = "modified stiffness factor"

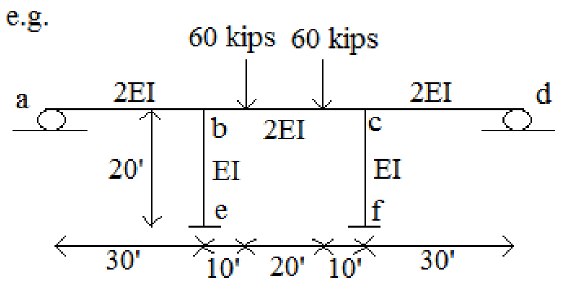
Pinned support : $M_{jb} = 3Ek_{jb}\theta = 4Ek_{jb}'\theta$, where $k_{jb}' = \frac{3}{4}k_{jb}$

Symmetrical : $M_{jc} = 2Ek_{jc}\theta = 4Ek_{jc}'\theta$, where $k_{jc}' = \frac{1}{2}k_{jc}$

Antisymmetrical : $M_{jd} = 6Ek_{jd}\theta = 4Ek_{jd}'\theta$, where $k_{jd}' = \frac{3}{2}k_{jd}$

$$M_{ja} = \frac{k_{ja}'}{\sum k'} M \qquad M_{jb} = \frac{k_{jb}'}{\sum k'} M \qquad M_{jc} = \frac{k_{jc}'}{\sum k'} M \qquad M_{jd} = \frac{k_{jd}'}{\sum k'} M$$

e.g.

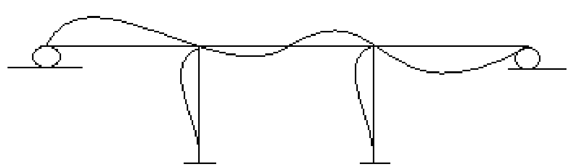


$$FEM_{bc} = FEM_{cb} = (\frac{10}{40})^2(\frac{30}{40})(60)(40)$$

$$-(\frac{10}{40})(\frac{30}{40})^2(60)(40) = -225kips$$

$$k_{ba} = \frac{2I}{30} = \frac{I}{15} \qquad k_{bc} = \frac{2I}{40} = \frac{I}{20} = k_{cb}$$

$$k_{cd} = \frac{2I}{30} = \frac{I}{15} \qquad k_{be} = \frac{I}{20} \qquad k_{cf} = \frac{I}{20}$$



modified stiffness:

$k_{ba}' = \frac{3}{4} k_{ba} = .05I = k_{cd}'$ *(pinned/roller support)*

$k_{bc}' = \frac{3}{2} k_{bc} = .075I = k_{cb}'$ *(antisymmetrical)*

$k_{be}' = k_{be} = .05I = k_{cf}'$

	ab	eb	be	ba	bc	cb	cd	cf	fc	dc
$\frac{k'}{\Sigma k}$	1	0	.2857	.2857	.4286	.4286	.2857	.2857	0	1
FEM					-225	-225				
DM			64.3	64.3	96.4	96.4	64.3	64.3		
COM		32.15							32.15	
Σ	0	32.15	64.3	64.3	-128.6	-128.6	64.3	64.3	32.15	0

(kips * ft)

note: only one cycle needed for this problem with modified stiffness approach

note: since the frame is symmetrical, we don't really need to tabularize all of the moments, but rather just half of the frame