coggle

Modules

Definitions

$A-$ Module M
i) M is an abelian group
ii) A is a unitary ring
iii) $a(v_1+v_2)=av_1+av_2$
and $(a_1+a_2)v=a_1v+a_2v$
iv) $a_1(a_2v)=(a_1a_2)v$
v) $1v=v$

Vector Space V
i) $K-$ module V
ii) K is a field

Homomorphism of $A-$ Modules
i) $\phi:M_1\longrightarrow M_2$
ii) $\phi(v_1+v_2)=\phi(v_1)+\phi(v_2)$
iii) $\phi(av)=a\phi(v)$

Monomorphism - Injective
Epimorphism - Surjective
Isomorphism - Bijective
Endomorphism - $M\longrightarrow M$
Automorphism - Bijective endomorphism

Submodule N of M
i) $N\leq M$
ii) N is closed for multiplication

Quotient Module M/N
i) M/N as groups
ii) $a(v+N)=av+N$

Results

Trivial Results

$a0_M=0_M$
$0_Rv=0_M$
$(-a)v=-(av)=a(-v)$
$n(av)=a(nv)=(na)v$

Isomorphism Theorems

First
IF
1- $\phi:M_1\longrightarrow M_2$ epimorphism
2- N is the kernel of ϕ
THEN
1- $M_1/N\simeq M_2$
2- There is a unique isomorphism $\bar{\phi}:M_1/N\longrightarrow M_2$ such that $\bar{\phi}(\underline{v})=\phi(v)$

Second
IF
1- B,C submodules of A
THEN
1- $\frac{B+C}{C}\simeq\frac{B}{B\cap C}$

Third
IF
1- B,C submodules of A
2- $C\subset B$
THEN
1- B/C is a submodule of A/C
2- $\frac{A/C}{B/C}\simeq\frac{A}{B}$

Theorems

Finite Groups Classification Theorems

Propositions

Important Examples

Algebra

Groups

Definitions

Group $(G,*)$
i) Associativity
ii) Identity
iii) Symetric Element

Abelian Group $(G,+)$
iv) Commutative

Homomorphism $\phi:G\longrightarrow H$
i) $\phi(g_1g_2)=\phi(g_1)\phi(g_2)$

Monomorphism - Injective
Epimorphism - Surjective
Isomorphism - Bijective
Endomorphism - $G\longrightarrow G$
Automorphism - Bijective endomorphism

Kernel K of $\phi:G\longrightarrow H$
i) $K=\{g\in G:\phi(g)=1_H\}$

Subgroup $H\leq G$
i) $H\subset G$
ii) H closed

Normal Subgroup $N\lhd G$
i) $N\leq G$
ii) $gng^{-1}\in N$

Equivalent Definitions of $N\lhd G$
i) $gNg^{-1}\subset N$
ii) $gNg^{-1}=N$
iii) $gN=Ng$
iv) {Left Cosets} = {Right Cosets}
v) N is the kernel of some endomorphism

$p-$Group
Any $g\in G$ has $|g|=p^k$

Sylow $p-$Subgroup
A maximal $p-$subgroup

Left Coset of H
$gH=\{gh:h\in H\}$
Right Coset of H
$Hg=\{hg:h\in H\}$

$[G:H]$ is the number of cosets of H

Quotient Group G/H
i) $H\lhd G$
ii) $G/H=\{gH:g\in G\}$
iii) $g_1(g_2H)=(g_1g_2)H$

Center of G
$C(G)=\{a\in G:za=az \text{ for every } z\in G\}$

Trivial Results

The identity is unique
The symmetric element is unique
$\phi(1_G)=1_H$
$\phi(g^{-1})=\phi(g)^{-1}$
$\phi(G)\leq H$

Particular Types of Groups

Torsion Subgroup

Torsion Free Group
Torsion Group
Solvable
Nilpotent
Cyclic
Free
Simple
Finitely Generated

Theorems

Isomorphism Theorems

First
IF
1- $\phi:G\longrightarrow H$ is an epimorphism
2- N is the kernel of ϕ
THEN
1- $G/N\simeq H$
2- There is a unique isomorphism $\bar{\phi}:G/N\longrightarrow H$ such that $\bar{\phi}(\underline{x})=\phi(x)$ for any $x\in G$

Second
IF
1- N and H are subgroups of G
2- $N\lhd G$
THEN
1- $HN\leq G$
2- $N\cap H\lhd H$
3- $\frac{H}{N\cap H}\simeq\frac{HN}{N}$

Third
IF
1- K and H normal subgroups of G
2- $K\subset H$
THEN
1- $K\lhd H$
2- $H/K\lhd G/K$
3- $\frac{G/K}{H/K}\simeq G/H$

Langrange Theorem

IF
1- $H\leq G$
THEN
1- $|G|=[G:H]|H|$

Sylow Theorems

First
IF
1- $|G|=p^nm$
2- $p\nmid m$
THEN
1- There is $H\leq G$ with $|H|=p^i$ for each $1\leq i\leq n$
2- $\forall\,|H_i|=p^i$, with $i<n$, then there is H_{i} such that $|H_i|=p^{i+1}$ and $H_i\lhd H_{i}$

Second
IF
1- H is a $p-$subgroup of a finite G
2- P is a Sylow $p-$subgroup of G
THEN
1- There is $x\in G$ such that $H\leq xPx^{-1}$
2- In particular, any two Sylow $p-$subgroup of G are conjugate

Third
IF
1- G finite
2- p prime
THEN
1- The number of Sylow $p-$subgroups divides $|G|$
2- The number of Sylow $p-$subgroups is of the form $kp+1$, with $k\geq 0$

Propositions

All cyclic groups are abelian
All cyclic groups are isomorphic to $\mathbb{Z}_p$, for some p prime

Important Examples

$(\mathbb{Z}_n,+)$ $(\mathbb{Z}_{mn},+)$ $(\mathbb{Z}_{p^k},+)$

Symmetric Group $A(S)$
Set of all the permutations of S

Alternate Group A_n
Set of even permutations of S

General Linear Group $GL(n,\mathbb{R})$

Orthogonal Group
Unitary Group

Rings

Definitions

Ring $(R,+,\times)$
i) $(R,+)$ is an abelian group
ii) $\times$ is associative
iii) $\times$ is distributive

Homomorphism $\phi:R\longrightarrow S$
i) $\phi(r_1+r_2)=\phi(r_1)+\phi(r_2)$
ii) $\phi(r_1\times r_2)=\phi(r_1)\times\phi(r_2)$

Monomorphism - Injective
Epimorphism - Surjective
Isomorphism - Bijective
Endomorphism - $R\longrightarrow R$
Automorphism - Bijective endomorphism

Subring $S\subseteq R$
i) $S\subset R$
ii) S is closed for $+$ and $\times$

Ideal $I\lhd R$
i) $I\lhd R$
ii) $\forall\, i\in I$

Prime
Maximal
Principal

Quotient Ring R/S
i) $S\lhd R$
ii) $s(r+S)=sr+S$

Types of Rings

Unitary
i) Has identity

Abelian
i) $\times$ is commutative

Domain
i) Doesn't have zero divisors

Principal Ideal Domain

Integral Domain
i) Domain
ii) Unitary
iii) Abelian

Euclidean Domain E
i) E is an Euclidian ring
ii) E is an integral domain

Euclidean Ring R
1- R is commutative
2- There is a function $\varphi:(R-\{0\})\longrightarrow\mathbb{N}$ such that
2a) If $a\,\vdots\,b\neq 0$ then $\varphi(a)\leq\varphi(ab)$
2b) If $b\neq 0$ then there exists $q,r\in R$ such that $a=qb+r$ with $r=0$ or $r\neq 0$ and $\varphi(r)<\varphi(b)$

Division Ring
i) Unitary
ii) Each element has inverse

Field
i) Has identity
ii) Is abelian
iii) Each element has inverse

Results

Trivial Results

$a\times 0=0$

Isomorphism Theorems

First
IF
1- $\phi:A\longrightarrow B$ is an epimorphism
2- I is the kernel of ϕ
THEN
1- $A/I\simeq B$
2- There is a unique isomorphism $\bar{\phi}:A/I\longrightarrow B$ such that $\bar{\phi}(\underline{a})=\phi(a)$

Second
IF
1- $I\lhd A$
2- $B\subseteq A$
THEN
1- $I+B\subseteq A$
2- $I\lhd I+B$
3- $I\cap B\lhd B$
4- $\frac{I+B}{I}\simeq\frac{B}{I\cap B}$

Third
IF
1- I and J ideals of A
2- $I\subset J$
THEN
1- $I\lhd J$
2- $J/I\lhd A/I$
3- $\frac{A/I}{J/I}\simeq\frac{A}{J}$

Theorems

Propositions

Important Examples