2

§21.2. THE HILBERT ACTION PRINCIPLE AND THE PALATINI METHOD OF VARIATION

Five days before Einstein presented his geometrodynamic law in its final and now standard form, Hilbert, animated by Einstein's earlier work, independently discovered (1915a) how to formulate this law as the consequence of the simplest action principle of the form (21.2-21.3) that one can imagine:

Variational principle the simplest route to Einstein's equation

$$L_{\text{geom}} = (1/16\pi)^{(4)}R. \tag{21.13}$$

(Replace $1/16\pi$ by $c^3/16\pi G$ when going from the present geometric units to conventional units; or divide by $\hbar \sim L^{*2}$ to convert from dynamic phase, with the units of action, to actual phase of a wave function, with the units of radians). Here ${}^{(4)}R$ is the four-dimensional scalar curvature invariant, as spelled out in Box 8.4.

This action principle contains second derivatives of the metric coefficients. In contrast, the action principle for mechanics contains only first derivatives of the dynamic variables; and similarly only derivatives of the type $\partial A_\alpha/\partial x^\beta$ appear in the action principle for electrodynamics. Therefore one might also have expected only first derivatives, of the form $\partial g_{\mu\nu}/\partial x^\gamma$, in the action principle here. However, no scalar invariant lets itself be constructed out of these first derivatives. Thus, to be an invariant, L_{geom} has to have a value independent of the choice of coordinate system. But in the neighborhood of a point, one can always so choose a coordinate system that all first derivatives of the $g_{\mu\nu}$ vanish. Apart from a constant, there is no scalar invariant that can be built homogeneously out of the metric coefficients and their first derivatives.

When one turns from first derivatives to second derivatives, one has all twenty distinct components of the curvature tensor to work with. Expressed in a local inertial frame, these twenty components are arbitrary to the extent of the six parameters of a local Lorentz transformation. There are thus $20 - 6 = 14$ independent local features of the curvature ("curvature invariants") that are coordinate-independent, any one of which one could imagine employing in the action principle. However, ${}^{(4)}R$ is the only one of these 14 quantities that is linear in the second derivatives of the metric coefficients. Any choice of invariant other than Hilbert's complicates the geometrodynamic law, and destroys the simple correspondence with the Newtonian theory of gravity (Chapter 17).

Scalar curvature invariant the only natural choice

Hilbert originally conceived of the independently adjustable functions of x, y, z, t in the variational principle as being the ten distinct components of the metric tensor in contravariant representation, $g^{\mu\nu}$. Later Palatini (1919) discovered a simpler and more instructive listing of the independently adjustable functions: not the ten $g^{\mu\nu}$ alone, but the ten $g^{\mu\nu}$ plus the forty $\Gamma^\alpha_{\mu\nu}$ of the affine connection.

To give up the standard formula for the connection Γ in terms of the metric g and let Γ "flap in the breeze" is not a new kind of enterprise in mathematical physics. Even in the simplest problem of mechanics, one can give up the standard formula for the momentum p in terms of a time-derivative of the coordinate x and also let