Equations:

$$\frac{\partial v(s,n)}{\partial n} + \frac{\partial u(s,n)}{\partial s} + \frac{\partial \xi(s,n)}{\partial s} + An\frac{dc(s)}{ds} = 0 \tag{1}$$

$$A_1\frac{\partial \xi(s,n)}{\partial n} + \frac{\partial v(s,n)}{\partial s} - c(s) + A_2 v(s,n) + A_3 c(s) = 0 \tag{2}$$

$$\frac{\partial u(s,n)}{\partial s} + 2A_2 u(s,n) = A_2(\xi(s,n) + Anc(s)) - A_1\frac{\partial \xi(s,n)}{\partial s} - A_2 nc(s) \tag{3}$$

Unknowns: $u(s,n), v(s,n), \xi(s,n)$

Boundary conditions: $v(s,1) = v(s,-1) = 0$

I am trying to solve the equations in this way:

a. Simplify equation (2) and come up with an approximate solution for ξ i.e:

$A_1\frac{\partial \xi(s,n)}{\partial n} = c(s) - A_3 c(s)$ Thus

$$\boxed{\xi(s,n) = Bc(s)n + f(s)} \tag{4}$$

b. Substitute (4) into equation (3) and solve for u i.e:

$$\frac{\partial u(s,n)}{\partial s} + 2A_2 u(s,n) = A_2\big(Bc(s)n + f(s) + Anc(s)\big) - A_1\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right) - A_2 nc(s)$$

Thus,

$$\frac{\partial u(s,n)e^{2A_2 s}}{\partial s} = A_2 e^{2A_2 s}\big(Bc(s)n + f(s) + Anc(s)\big) - A_1 e^{2A_2 s}\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right) - A_2 e^{2A_2 s} nc(s)$$ and therefore,

$$\boxed{u(s,n) = A_2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\big(Bc(s)n + f(s) + Anc(s)\big)\,ds - A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right)ds - A_2 n e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds + g(n)e^{-2A_2 s}} \tag{5}$$

c. Substitute (4) and (5) into equation (1) to solve for v

$$\frac{\partial v(s,n)}{\partial n} = 2A_2 u(s,n) - A_2\big(Bc(s)n + f(s) + Anc(s)\big) + (A_1 - 1)\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right) + A_2 nc(s) - An\frac{dc(s)}{ds}$$

$$\frac{\partial v(s,n)}{\partial n} =$$

$$2{A_2}^2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\big(Bc(s)n + f(s) + Anc(s)\big)\,ds - 2A_2A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right)ds - 2{A_2}^2 n e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds + 2A_2 g(n)e^{-2A_2 s} - A_2\big(Bc(s)n + f(s) + Anc(s)\big) + (A_1 - 1)\left(Bn\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right) + A_2 nc(s) - An\frac{dc(s)}{ds}$$

Thus

$$\boxed{\begin{aligned} &v(s,n) = \\ &2{A_2}^2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bc(s)\frac{n^2}{2} + f(s)n + A\frac{n^2}{2}c(s)\right)ds - \\ &2A_2A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(B\frac{n^2}{2}\frac{dc(s)}{ds} + \frac{df(s)}{ds}n\right)ds - 2{A_2}^2\frac{n^2}{2}e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds + \\ &2A_2 e^{-2A_2 s}\int_{-1}^{n} g(n)\,dn - A_2\left(Bc(s)\frac{n^2}{2} + f(s)n + A\frac{n^2}{2}c(s)\right) + (A_1 - 1)\left(B\frac{n^2}{2}\frac{dc(s)}{ds} + \right. \\ &\left. n\frac{df(s)}{ds}\right) + A_2\frac{n^2}{2}c(s) - A\frac{n^2}{2}\frac{dc(s)}{ds} + h(s) \qquad (6)\end{aligned}}$$

d. Apply boundary conditions and solve for arbitrary functions

$$v(s,-1) =$$

$$2{A_2}^2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bc(s)\frac{1}{2} - f(s) + A\frac{1}{2}c(s)\right)ds - 2A_2A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(B\frac{1}{2}\frac{dc(s)}{ds} - \frac{df(s)}{\partial s}\right)ds - {A_2}^2 e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds - A_2\left(Bc(s)\frac{1}{2} - f(s) + A\frac{1}{2}c(s)\right) + (A_1 - 1)\left(B\frac{1}{2}\frac{dc(s)}{ds} - \frac{df(s)}{\partial s}\right) + A_2\frac{1}{2}c(s) - A\frac{1}{2}\frac{dc(s)}{ds} + h(s) = 0 \qquad (*)$$

$$v(s,1) =$$

$$2{A_2}^2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bc(s)\frac{1}{2} + f(s) + A\frac{1}{2}c(s)\right)ds - 2A_2A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(B\frac{1}{2}\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right)ds - {A_2}^2 e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds + 2A_2 e^{-2A_2 s}\int_{-1}^{1} g(n)\,dn - A_2\left(Bc(s)\frac{1}{2} + f(s) + A\frac{1}{2}c(s)\right) + (A_1 - 1)\left(B\frac{1}{2}\frac{dc(s)}{ds} + \frac{df(s)}{ds}\right) + A_2\frac{1}{2}c(s) - A\frac{1}{2}\frac{dc(s)}{ds} + h(s) = 0 \qquad (**)$$

(*) – (**)

$$-4A_2{}^2e^{-2A_2s}\int_0^s f(s)e^{2A_2s}\,ds + 4A_2A_1e^{-2A_2s}\int_0^s \frac{df(s)}{ds}e^{2A_2s}\,ds - A_2e^{-2A_2s}\int_{-1}^1 g(n)\,dn +$$

$$2A_2f(s) - 2(A_1-1)\frac{df(s)}{ds} = 0$$

Rewriting

$$-4A_2{}^2\int_0^s f(s)e^{2A_2s}\,ds + 4A_2A_1\int_0^s\left(\frac{df(s)e^{2A_2s}}{ds} - 2A_2f(s)e^{2A_2s}\right)ds - A_2\int_{-1}^1 g(n)\,dn +$$

$$2A_2f(s)e^{2A_2s} - 2(A_1-1)\left(\frac{df(s)e^{2A_2s}}{ds} - 2A_2f(s)e^{2A_2s}\right) = 0$$

Differentiating w.r.t s

$$-2A_2{}^2f(s)e^{2A_2s} + 2A_2A_1\left(\frac{df(s)e^{2A_2s}}{ds} - 2A_2f(s)e^{2A_2s}\right) + A_2\frac{df(s)e^{2A_2s}}{ds} - (A_1 -$$

$$1)\left(\frac{d^2f(s)e^{2A_2s}}{ds^2} - 2A_2\frac{df(s)e^{2A_2s}}{ds}\right) = 0$$

Let Q = $f(s)e^{2A_2s}$, rearranging

$$(A_1-1)\frac{d^2Q}{ds^2} - A_2(4A_1-1)\frac{dQ}{ds} + 2A_2{}^2(2A_1+1)Q = 0 \qquad (7)$$

This second order ODE has a solution of the form:

Q = $C_1e^{r_1s} + C_2e^{r_2s}$

Where r_1 and r_2 are the roots of the characteristic equation:

$r_1 = \frac{A_2(2A_1+1)}{(A_1-1)}$ and $r_2 = 2A_2$

Thus,

$$f(s)e^{2A_2s} = C_1e^{\frac{A_2(2A_1+1)}{(A_1-1)}s} + C_2e^{2A_2s}$$

$$f(s) = C_1e^{\frac{3A_2}{(A_1-1)}s} + C_2 \qquad (8)$$

Hence

$$\int_{-1}^1 g(n)\,dn = -4A_2\int_0^s f(s)e^{2A_2s}\,ds + 4A_1\int_0^s\left(\frac{df(s)e^{2A_2s}}{ds} - 2A_2f(s)e^{2A_2s}\right)ds +$$

$$2f(s)e^{2A_2s} - 2\frac{(A_1-1)}{A_2}\left(\frac{df(s)e^{2A_2s}}{ds} - 2A_2f(s)e^{2A_2s}\right) \qquad (9)$$

and

$$
\begin{aligned}
h(s) = {}& -2{A_2}^2 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(Bc(s)\frac{1}{2} - f(s) + A\frac{1}{2}c(s)\right)ds + 2A_2A_1 e^{-2A_2 s}\int_0^s e^{2A_2 s}\left(B\frac{1}{2}\frac{dc(s)}{ds} - \frac{df(s)}{\partial s}\right)ds \\
& + {A_2}^2 e^{-2A_2 s}\int_0^s c(s)e^{2A_2 s}ds + A_2\left(Bc(s)\frac{1}{2} - f(s) + A\frac{1}{2}c(s)\right) - (A_1 - 1)\left(B\frac{1}{2}\frac{dc(s)}{ds} - \frac{df(s)}{\partial s}\right) - A_2\frac{1}{2}c(s) + A\frac{1}{2}\frac{dc(s)}{ds} \qquad (10)
\end{aligned}
$$

e. Modify the approximation of ξ with (2) so that

$$A_1\frac{\partial\xi(s,n)}{\partial n} = -\frac{\partial v(s,n)}{\partial s} + c(s) - A_2v(s,n) - A_3c(s)$$

$$\xi(s,n) = -\frac{1}{A_1}\int_{-1}^{n}\left(\frac{\partial v(s,n)}{\partial s} + A_2v(s,n)\right)dn + Bc(s)n + f_1(s)$$

f. Substitute in (3) and compute new u

g. Substitute new u in (1) and compute new v

h. Continue loop until…