Appendix B

BAC-CAB Formulas

Using the python geometric algebra module *GA*[1] several formulas containing the dot and wedge products can be reduced. Let a, b, c, d, and e be vectors, then we have

$$
\begin{aligned}
a\cdot(bc) &= (b\cdot c)\,a-(a\cdot c)\,b+(a\cdot b)\,c && \text{(B.1)}\\
a\cdot(b\wedge c) &= (a\cdot b)\,c-(a\cdot c)\,b && \text{(B.2)}\\
a\cdot(b\wedge c\wedge d) &= (a\cdot d)\,(b\wedge c)-(a\cdot c)\,(b\wedge d)\\
&\quad+(a\cdot b)\,(c\wedge d) && \text{(B.3)}\\
a\cdot(b\wedge c\wedge d\wedge e) &= -\,(a\cdot e)\,(b\wedge c\wedge d)+(a\cdot d)\,(b\wedge c\wedge e)\\
&\quad-(a\cdot c)\,(b\wedge d\wedge e)+(a\cdot b)\,(c\wedge d\wedge e) && \text{(B.4)}\\
(a\cdot(b\wedge c))\cdot(d\wedge e) &= ((a\cdot c)\,(b\cdot e)-(a\cdot b)\,(c\cdot e))\,d\\
&\quad+((a\cdot b)\,(c\cdot d)-(a\cdot c)\,(b\cdot d))\,e. && \text{(B.5)}
\end{aligned}
$$

If in equation B.2 the vector b is replace by a vector differential operator such as ∇, ∂, or D (we will use D as an example) it can be rewritten as

$$
\begin{aligned}
(a\cdot D)\,c &= a\cdot(D\wedge c)+(a\cdot\dot{c})\,\dot{D}\\
&= a\cdot(D\wedge c)+\dot{D}\,(a\cdot\dot{c})\\
&= a\cdot(D\wedge c)+\dot{D}\,(\dot{c}\cdot a) && \text{(B.6)}
\end{aligned}
$$

Cyclic reduction formulas are

$$a\cdot(b\wedge c)+c\cdot(a\wedge b)+b\cdot(c\wedge a)=0 \qquad \text{(B.7)}$$

[1]Alan Macdonald website `http://faculty.luther.edu/~macdonal/vagc/index.html`

$$a\,(b\wedge c) - b\,(a\wedge c) + c\,(a\wedge b) = 3a\wedge b\wedge c \tag{B.8}$$

$$\begin{aligned} a\,(b\wedge c\wedge d) - b\,(a\wedge c\wedge d) + c\,(a\wedge b\wedge d) - d\,(a\wedge b\wedge c) = \\ 4a\wedge b\wedge c\wedge d \end{aligned} \tag{B.9}$$

Basis blade reduction formula

$$\begin{aligned} (a\wedge b)\cdot(c\wedge d) &= ((a\wedge b)\cdot c)\cdot d \\ &= (a\cdot d)\,(b\cdot c) - (a\cdot c)\,(b\cdot d) \end{aligned} \tag{B.10}$$

But we also have that

$$((a\wedge b)\cdot c)\cdot d = (a\cdot d)\,(b\cdot c) - (a\cdot c)\,(b\cdot d) \tag{B.11}$$

which gives the same results as equation B.10. Since for any bivector blade $B = a\wedge c$ we have

$$(B\cdot c)\cdot d = B\cdot(c\wedge d)\,. \tag{B.12}$$

By linearity equation B.12 is also true for any bivector B since B is a superposition of bivector blades.

Finally one formula for reducing the commutator product of two bivectors

$$\begin{aligned} (a\wedge b)\times(c\wedge d) &= (a\cdot d)\,b\wedge c - (a\cdot c)\,b\wedge d \\ &\quad + (b\cdot c)\,a\wedge d - (b\cdot d)\,a\wedge c \end{aligned} \tag{B.13}$$