Grand Unified Theories

Block Course of the International Graduate School, GRK 881, Sept. 2008 — S. Wiesenfeldt

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Standard Model Fields. The standard model of particle physics is based on the gauge group $G_{\rm SM} = SU(3)\times SU(2)\times U(1)_Y$. It contains twelve gauge bosons with spin 1: eight gluons of $SU(3)_C$, three $SU(2)_L$ weak bosons and the hypercharge boson of $U(1)_Y$,

gauge boson	gauge group	quantum numbers	coupling
gluons	$SU(3)_C$	(8,1,0)	g_3
W bosons	$SU(2)_L$	(1,3,0)	g_2
B boson	$U(1)_Y$	(1,1,0)	g'

The fundamental fermionic entities are three generations of quarks and leptons,

quark	quantum numbers	electric charge	lepton	quantum numbers	electric charge
$q = \binom{u_{\rm L}}{d_{\rm L}}$	$(3, 2, \frac{1}{6})$	$\begin{smallmatrix}+2/3\\ -1/3\end{smallmatrix}$	$\ell = \binom{\nu_{\rm L}}{e_{\rm L}}$	$(1, 2, -\frac{1}{2})$	$\begin{smallmatrix}0\\ -1\end{smallmatrix}$
$u^c_{\rm L}$	$(3^*, 1, -\frac{2}{3})$	$-2/3$			
$d^c_{\rm L}$	$(3^*, 1, \frac{1}{3})$	$+1/3$	$e^c_{\rm L}$	$(1, 1, 1)$	$+1$

We will identify the multiplets by their quantum numbers. Each fermion family is given by the sum

$$(3, 2, \tfrac{1}{6}) \oplus (3^*, 1, -\tfrac{2}{3}) \oplus (3^*, 1, \tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2}) \oplus (1, 1, 1)\ , \tag{1}$$

the gauge bosons by

$$(8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0)\ . \tag{2}$$

There is no left-handed antineutrino in the standard model which would be neutral with respect to all interactions, i.e. (1,1,0).

The fermions are listed in terms of left-chiral spinors. In four-component notation, one should think of

$$u_{\rm L} = \begin{pmatrix}\xi\\ 0\end{pmatrix}, \qquad (u^c)_{\rm L} = \begin{pmatrix}\chi\\ 0\end{pmatrix}, \tag{3}$$

where ξ and χ are $(\frac{1}{2}, 0)$ Weyl spinors. A priori $u_{\rm L}$ and $(u^c)_{\rm L}$ are totally independent; however, they will eventually pair up to make a Dirac spinor

$$u = \begin{pmatrix}\xi\\ \epsilon\chi^*\end{pmatrix}. \tag{4}$$

Pre-Exercise: Check that the standard model is anomaly-free.

The Generators of SU(5). The group $SU(n)$ is defined by its fundamental representation, the group of $n\times n$ unitary matrices with determinant one. A general transformation can be written as

$$U = \exp\left\{-i\sum_{j=1}^{n^2-1} \alpha^j L^j\right\}\ , \tag{5}$$

where the generators $L^j = \frac{1}{2}\lambda^j$ are Hermitean and traceless and normalized so that $\operatorname{tr}(L^i L^j) = \frac{1}{2}\delta^{ij}$. In case of SU(2) and SU(3), the λ^j correspond to the Pauli and Gell-Mann matrices, respectively; for SU(5), they are given as follows. The first eight matrices correspond to SU(3),

$$\lambda_1 = \begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix}0&-i&0&0&0\\i&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix}1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix}0&0&1&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_5 = \begin{pmatrix}0&0&-i&0&0\\0&0&0&0&0\\i&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_6 = \begin{pmatrix}0&0&0&0&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix}0&0&0&0&0\\0&0&-i&0&0\\0&i&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&-2&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}; \tag{6a}$$

the following twelve matrices to the broken generators,

$$\lambda_9 = \begin{pmatrix}0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_{10} = \begin{pmatrix}0&0&0&-i&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_{11} = \begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix},$$

$$\lambda_{12} = \begin{pmatrix}0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\end{pmatrix}, \quad \lambda_{13} = \begin{pmatrix}0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_{14} = \begin{pmatrix}0&0&0&0&0\\0&0&0&-i&0\\0&0&0&0&0\\0&i&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\lambda_{15} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix}, \quad \lambda_{16} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0\end{pmatrix}, \quad \lambda_{17} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix},$$

$$\lambda_{18} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&-i&0\\0&0&i&0&0\\0&0&0&0&0\end{pmatrix}, \quad \lambda_{19} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&1&0&0\end{pmatrix}, \quad \lambda_{20} = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0\end{pmatrix}; \tag{6b}$$

and the remaining matrices to SU(2) and hypercharge,

$$\lambda_{21} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}, \qquad \lambda_{22} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&i&0 \end{pmatrix}, \qquad \lambda_{23} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&-1 \end{pmatrix},$$

$$\lambda_{24} = \frac{1}{\sqrt{15}}\,\mathrm{diag}\,(-2,-2,-2,3,3) \tag{6c}$$

Since we embed a group of the form $\mathrm{SU}(n)\times\mathrm{SU}(m)\times\mathrm{U}(1)$ into $\mathrm{SU}(n+m)$, we choose $\mathrm{SU}(n)$ to act on the first n indices and $\mathrm{SU}(m)$ on the last m indices. Both of these subgroups commute with the U(1) which we can take to be m on the first n indices and $-n$ on the last m.

Gauge Fields. There are n^2-1 Hermitean gauge fields A^j. We define the $n\times n$ matrix A,

$$A \equiv \sqrt{2}\sum L^j A^j \; ; \tag{7}$$

for SU(5), it explicitly reads

$$A = \left(\begin{array}{ccc|cc} G^1_1 - \frac{2B}{\sqrt{30}} & G^1_2 & G^1_3 & \bar{X}^1 & \bar{Y}^1 \\ G^2_1 & G^2_2 - \frac{2B}{\sqrt{30}} & G^2_3 & \bar{X}^2 & \bar{Y}^2 \\ G^3_1 & G^3_2 & G^3_3 - \frac{2B}{\sqrt{30}} & \bar{X}^3 & \bar{Y}^3 \\ \hline X_1 & X_2 & X_3 & \frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & \frac{W_1+iW_2}{\sqrt{2}} \\ Y_1 & Y_2 & Y_3 & \frac{W_1-iW_2}{\sqrt{2}} & -\frac{W_3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{array}\right). \tag{8}$$

The entries represent the gauge bosons transforming according to the adjoint representation,

$$\mathsf{24} \to (8,1,0)\oplus(1,3,0)\oplus(1,1,0)\oplus(3,2^*,-\tfrac{5}{6})\oplus(3^*,2,\tfrac{5}{6})\;. \tag{9}$$

We identify the SM gauge bosons (2) and find twelve new ones, the X and Y bosons.

1. Verify that the presence of the X and Y bosons above M_{GUT} ensures that the one-loop coefficients of the beta-functions of the standard model, b_i, coincide.

Fermion Representations. We can group each SM fermion generation (1) into $\mathsf{5}^*\oplus\mathsf{10}$ of SU(5). Ignoring mixings, we get

$$\mathsf{5}^* = \left(\begin{array}{c} d^c_1 \\ d^c_2 \\ d^c_3 \\ \hline e^- \\ -\nu_e \end{array}\right), \qquad \mathsf{10} = \frac{1}{\sqrt{2}}\left(\begin{array}{ccc|cc} 0 & u^c_3 & -u^c_2 & -u^1 & -d^1 \\ -u^c_3 & 0 & u^c_1 & -u^2 & -d^2 \\ u^c_2 & -u^c_1 & 0 & -u^3 & -d^3 \\ \hline u^1 & u^2 & u^3 & 0 & -e^c \\ d^1 & d^2 & d^3 & e^c & 0 \end{array}\right). \tag{10}$$

2. By means of $\mathsf{5}\to(3,1,-\frac{1}{3})\oplus(1,2,+\frac{1}{2})$, compute the decompositions of the ten and fifteen-dimensional representations of SU(5) with respect to G_{SM}.
3. Check that SU(5) with three generations of $\mathsf{5}^*\oplus\mathsf{10}$ is anomaly free.
4. *(a)* Write down the Yukawa couplings of the minimal SU(5) model in standard model fields. In view of the supersymmetric model, consider the quintet fields, $H(5) = (H_C, H_u)$ and $\overline{H}(\mathsf{5}^*) = \big(\overline{H}_C, H_d\big)$ as independent.
(b) Integrating out the heavy color-triplets, derive the $LLLL$ and $RRRR$ operators for proton decay.

5. In the minimal SU(5) model, the Lagrangian possesses a global $\mathrm{U}(1)_X$ symmetry, which is spontaneously broken by the vev of the quintet H.
(a) Assign the charges to the various fields, starting with $Q_H = -2$.
(b) Show that a linear combination of $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_X$ remains unbroken and no Goldstone bosons appears after electroweak symmetry breaking. Interpret this global symmetry.

Alternative Route: Left-Right and Pati-Salam Symmetry, SO(10)

Left-Right Model:	$Q = \binom{u_{\mathrm{L}}}{d_{\mathrm{L}}}$	$(3,2,1,\frac{1}{3})$	$L = \binom{\nu_{e\mathrm{L}}}{e^-_{\mathrm{L}}}$	$(1,2,1,-1)$
	$Q^c = \binom{d^c_{\mathrm{L}}}{u^c_{\mathrm{L}}}$	$(3^*,1,2,\frac{1}{3})$	$L^c = \binom{\nu^c_{\mathrm{L}}}{e^c_{\mathrm{L}}}$	$(1,1,2,1)$

Pati-Salam Model:	$F = (Q,L)$	$(4,2,1)$
	$F^c = (Q^c,L^c)$	$(4^*,1,2)$

6. The breaking $\mathrm{SO}(10)\to\mathrm{SU}(5)\times\mathrm{U}(1)_X$ does not uniquely lead to the Georgi-Glashow model. There is an alternative solution, called *flipped SU(5)*.
(a) Discuss how the SM model fermions can be embedded into
$$\mathrm{SO}(10)\to\mathrm{SU}(5)\times\mathrm{U}(1)_X\to\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Z\times\mathrm{U}(1)_X$$
and specify how hypercharge and $B-L$ are given in terms of the Z and X charges.
(b) A pair of ten-dimensional Higgs fields breaks flipped SU(5) to the SM. Check that hypercharge remains unbroken.
(c) Next, the electroweak symmetry is broken by a pair of quintets. Show that only the color-triplets of the quintets become massive via the couplings to the GUT Higgs fields and derive the fermion mass relations.

Supersymmetric GUTs For unification in the SM vs. the MSSM, see
`http://www-ekp.physik.uni-karlsruhe.de/~deboer/html/Forschung/unification_eng.eps`

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7. In supersymmetric SU(5), the $LLLL$ and $RRRR$ operators are of mass-dimension five and give the dominant contribution to the proton decay amplitude. We can transform the SM fields such that the couplings to the weak doublets are of the common form
$$W_{\mathrm{mass}} = q^\top\,\mathsf{D}_u\,u^c\,H_u + q^\top\,V_{\mathrm{CKM}}\mathsf{D}_d\,d^c\,H_d + e^{c\top}\,\mathsf{D}_d\,\ell\,H_d\;,$$
where D denotes the diagonalized mass matrices. Check that in this basis, the baryon number violating couplings read
$$W_{\not B} = \tfrac{1}{2}\,q^\top\,\mathsf{D}_u P\,q\,H_C + Q^\top\,V_{\mathrm{CKM}}\mathsf{D}_d\,\ell\,\overline{H}_C + u^{c\top}\,\mathsf{D}_u V^*_{\mathrm{CKM}}\,e^c\,H_C + u^{c\top}\,P^* V_{\mathrm{CKM}}\mathsf{D}_d\,d^c\,\overline{H}_C\;.$$
with an undetermined phase matrix P. [Use the results of Ex. 4(a), and recall that the up-type Yukawa coupling is symmetric.]