From Bose-Einstein to "correct Boltzmann counting" for indistinguishable particles?

The total number of arrangements of N indistinguishable particles among k energy levels with g_i states in level i such that n_i particles are in level i is:

$$W = \prod_{i=1}^{k} \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!}$$

A single factor in this product written without index becomes:

$$W = \frac{(n + g - 1)!}{n!\,(g - 1)!}$$

Assuming g >> n >> 1 and using Stirling's approximation ln x! ≈ x ln x – x in the form x! ≈ $x^x e^{-x}$ a few times and $e^x \approx 1 + x$ once gives:

$$W = \frac{(n + g - 1)!}{n!\,(g - 1)!} \approx \frac{(n + g)!}{n!\,g!} \approx \frac{(n + g)^{n+g}\,e^{-(n+g)}}{n^n e^{-n}\,g^g e^{-g}} = \frac{(n + g)^{n+g}}{n^n\,g^g} = \frac{g^n (n + g)^{n+g}}{n^n\,g^{n+g}}$$

$$= \frac{g^n \left(\frac{n}{g} + 1\right)^{n+g}}{n^n} \approx \frac{g^n \left(e^{\frac{n}{g}}\right)^{n+g}}{n^n} = \frac{g^n\,e^{\frac{n^2}{g}}\,e^n}{n^n} = \frac{g^n\,e^{\frac{n^2}{g}}}{n^n\,e^{-n}} \approx \frac{g^n}{n!} e^{\frac{n^2}{g}}$$