Ideas for the bouncing ball model

- ### Free fall

This bit is really simple, the only force is gravity. Therefore:

$$F = m\ddot{x} = mg \Longrightarrow \ddot{x} = g$$

$$\dot{x} = gt + v_0$$

$$x = \frac{1}{2}gt^2 + v_0t + x_0$$

Here I have defined down, i.e. towards the ground to be the positive direction. I'll stick to this throughout.

- ### Ball has hit the ground and is busy compressing

When the ball comes into contact with the ground, it experiences two additional forces: the one is a damping force that is always directed opposite to the velocity, the other is the spring force. For this part of the derivation, the spring force opposes motion as the ball is being compressed. The damping force always opposes motion. The equations are:

$$F = m\ddot{x} = mg - b\dot{x} - k(r - x)$$

$$\ddot{x} + \frac{b}{m}\dot{x} - \frac{k}{m}x = g - \frac{k}{m}r$$

Here, we define *k* to be the spring constant, *b* to be the damping coefficient and *r* to be the radius of the ball. Since the term on the right hand side of the equals is a constant, I shall replace it with *K*, yielding

$$\ddot{x} + \frac{b}{m}\dot{x} - \frac{k}{m}x = K$$

$$\ddot{x} + \frac{b}{m}\dot{x} - \frac{k}{m}x - K = 0$$

This differential equation can now be implemented using integrators and summers in Simulink.

The question now is how do you work out *k* and *b*. Since *k* is a spring constant, it can probably be worked out using the equipment at the Civil labs, which I think we have done. It is a little harder to work out *b* though.

The damping in the system represents an energy loss. Therefore, the energy before the ball collides with the floor will be greater than the energy when it leaves the floor. The energy is calculated from the velocity and therefore, the velocity after the collision must be smaller than the velocity before the collision. This doesn't help much as the velocities are difficult to determine.

If we go back to the equations for the free fall motion, one can see that it should be possible to determine the initial velocity v_0 if the maximum height that is achieved in that bounce is measured.

The problem with this approach is that we need to know the time that the ball took to reach that height.

This little problem can be circumvented when we observe that we can use the second and third equations presented under the free fall section. We would use $\dot{x} = 0$, $x_0 = r$ and $x = h$, where *h* is the height that the ball reached in that time. This leaves the time taken and the initial velocity (v_0). If we solve the equations simultaneously, then we should be able to get the initial velocity and eliminate time.

Hopefully, this initial velocity can be inserted into the equations for damped motion as a end condition. I am not sure how this will generate an answer though. This doesn't seem too promising though. The alternative is to investigate solving the differential equation for x. I'll use the same method we used in ETN2B.

The forced response is easy enough. First rewrite the equation:

$$\ddot{x} + \frac{b}{m}\dot{x} - \frac{k}{m}x = K$$

Then assume the solution for the forced response is x is constant. It is easy to see that the answer is

$$x = -\frac{m}{k}K = -\frac{m}{k}\left(g - \frac{k}{m}r\right) = r - \frac{m}{k}g = B$$

I have called it *B* because I need some constant to make future equations easy.

The natural response comes from using the differential operator s:

$$s^2 + \frac{b}{m}s - \frac{k}{m} = 0$$

$$ms^2 + bs - k = 0$$

The solutions for s are:

$$s = \frac{-b \pm \sqrt{b^2 + 4mk}}{2m}$$

Since *k* and *m* are always positive, we only have real unequal roots. This corresponds to the over damped case. But what do we do with this now? The form of the solution x is

$$x = A_1 e^{s_1 t} + A_2 e^{s_2 t} + B$$

$$\dot{x} = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$$

$$\ddot{x} = {s_1}^2 A_1 e^{s_1 t} + {s_2}^2 A_2 e^{s_2 t}$$

The next step is to use the initial conditions of the system (t=0) to calculate A_1 and A_2. The problem is that we also have to solve for b, giving us three variables and two equations. The initial position is known. The velocity is not precisely known, unless we try to measure it, but it can be deduced, like the final velocity is.

A possibility to investigate is using the other set of known points, viz. the final conditions. The time is not known but the velocity and position are known. This gives the small problem of adding two exponential functions to the whole mess.

Another possibility is to look at energy constraints. The damping force represents a loss in kinetic energy for the ball. Since, with the damping included, we have a closed system, the change in kinetic energy of the ball can only occur if some force does work on the ball. The spring force alone cannot cause this.

If we only had the spring force, the system would be a simple harmonic oscillator and the velocity entering the collision would equal the velocity leaving the collision. Therefore, the work done by the damping force must equal the change in velocity. I shall call the initial velocity $\dot{x}_0$. To simplify the integration, I will also only work with half of the journey, i.e. integrating from the start of the collision to the point where the velocity is zero, which is at r_0.

$$2\int_{r}^{r_0} -b\dot{x}\,dt = -\frac{1}{2}m\dot{x}_0{}^2$$

$$2\int_{r}^{r_0} b\dot{x}\,dt = \frac{1}{2}m\dot{x}_0{}^2$$

Now comes the questionable math.

$$\dot{x} = \frac{dx}{dt}$$

$$2\int_{r}^{r_0} b\dot{x}\,dt = 2\int_{r}^{r_0} b\frac{dx}{dt}\,dt = 2\int_{r}^{r_0} b\,dx = 2b(r_0 - r)$$

At this point, I am stumped by r_0 which would need to be measured. This is very difficult because the ball only comes into contact with the floor for a short period of time and, judging by how difficult it is to squash it, it probably doesn't deform all that much.

- **Ball has reached its maximum compression and starts expanding again**

At this point, the ball has zero velocity. The damping will continue to oppose motion but the spring effect in the ball will now aid motion. Therefore, gravity, as always, points down. Velocity and the spring force both point upwards. The same equations that were developed above can be used. The $-b\dot{x}$ term will now become positive, indicating a force pushing downwards, which is as was expected. The formulas that include damping are only used when $x \leq r$.