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Bell's theory with no locality assumption

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Abstract. We prove versions of the Bell and the GHZ theorems that do not assume locality but only the effect after cause principle (EACP) according to which for any Lorentz observer the value of an observable cannot change because of an event that happens after the observable is measured. We show that the EACP is strictly weaker than locality. As a consequence of our results, locality cannot be considered as the common cause of the contradictions obtained in all versions of Bell's theory. All versions of Bell's theorem assume weak realism according to which the value of an observable is well defined whenever the measurement could be made and some measurement is made. As a consequence of our results, weak realism becomes the only hypothesis common to the contradictions obtained in all versions of Bell's theory. Usually, one avoids these contradictions by assuming non-locality; this would not help in our case since we do not assume locality. This work indicates that it is weak realism, not locality, that needs to be negated to avoid contradictions in microscopic physics, at least if one refuses as false the de Broglie-Bohm hidden variable theory because of its essential violation of Lorentz invariance.

1 Introduction

Following Bohm 's version [1] of the EPR *gedanken experiment* [2], we consider entangled pairs of spin-$\frac{1}{2}$ particles such that the spin part of the wave function is the singlet state (at any location pair (x_1, x_2)):

$$\Psi\,(x_1, x_2) = \frac{1}{\sqrt{2}}\,(|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B)\,. \qquad (1)$$

This sum of tensor products represents an example of entanglement, which means that this expression cannot be rewritten as one tensor product of one particle states. We even have here a maximal entanglement since all the summands have identical statistical weights. The particles of the pair indexed by i are called $(p_A)_i$ and $(p_B)_i$. For each i the particle $(p_A)_i$ flies to Alice who is equipped with the measurement tool E while $(p_B)_i$ flies to Bob who handles the measurement tool P. E and P can be chosen as Stern-Gerlach magnets if, following Stapp [3] we use neutral particles, say neutrons. There is a source S of entangled pairs and, together with the tools E and P, the source S is attached to the laboratory frame; we assume that the measurements at E and P are (essentially) simultaneous in that frame.

Alice chooses the oriented axes $(a_A)_i$ and, with Spin(q) standing for the spin of particle q, she observes the sequence $\mathcal{E}_i$ of normalized projections of Spin$((p_A)_i)$ along $(a_A)_i$ while Bob chooses the oriented axes $(a_B)_i$ and observes the sequence $\mathcal{P}_i$ of normalized projections of Spin$((p_B)_i)$ along $(a_B)_i$. Bohm [1] noticed in particular that any observation $\sigma \in \{-1, +1\}$ by Alice along $(a_A)_i$ would necessarily correspond, because of the structure of the singlet state, to the observation $-\sigma$ by Bob if he would choose $(a_B)_i = (a_A)_i$. We will not recall nor revisit here the EPR paper, nor comment the way the authors themselves or Bell considered the issues raised in [2] or in [1]. The consideration of angles between the oriented axes $(a_A)_i$ and $(a_B)_i$ that can take any value in a setting that is otherwise the one proposed by Bohm is essential in the development of Bell's theory [4], and it is precisely this theory that we revisit here (Bohm used right or zero angles between axes in [1] as he was merely proposing a new version of the content of the EPR paper [2]). The experiment that consists in emitting successive pairs in the singlet state and measuring the normalized projections of the associated spins is repeated a large number of times in order to get statistically significant results. In order to achieve the same goal of significant statistics, the oriented axes $(a_A)_i$ and $(a_B)_i$ are usually kept constant for long sequences of values of i (on such stretches of constancy, one may suppress the index i). If the observation is $\mathcal{Q}_i$, we write $|\mathcal{Q}_i\rangle$ for the corresponding spin part of the state, and we denote by X the sequence with generic element X_i. Thus, the correlation $\langle \mathcal{U}, \mathcal{V}\rangle$ is equal to Dirac's braket $\langle \mathcal{U}|\mathcal{V}\rangle$ whenever both $|\mathcal{U}\rangle$ and $|\mathcal{V}\rangle$ are quantum mechanical states, but we will prefer the statistical notation. We denote by $\langle a_1; a_2\rangle$ the angle between the two oriented axes a_1 and a_2 and we associate to any oriented axis a_C the

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angle $\theta_C = \langle a_C; a_0 \rangle$ where a_0 is some oriented axis of reference that points, say, horizontally and to the right when looking from the far side along the departing particle flying toward Alice. Recall that quantum mechanics predicts probabilities of equality or equivalently correlations, the equivalence of the two viewpoints being captured in our case in the following identity:

$$\langle \mathcal{E}, \mathcal{P} \rangle = 2\text{Prob}\left(\mathcal{E}_i = \mathcal{P}_i\right) - 1 \tag{2}$$

where Prob(event) is the probability of that event (for a general reference for quantum mechanics covering somehow Bell's theory and in particular the GHZ theorem, see for instance [5] or [6]).

For the sequences $\mathcal{E}$ and $\mathcal{P}$ that we have defined for the spin-$\frac{1}{2}$ singlet state (1), quantum mechanics predicts what we call the *twisted Malus law* that differs from the usual Malus law by the minus sign:

$$\langle \mathcal{E}, \mathcal{P} \rangle = -\cos\left(\theta_A - \theta_B\right). \tag{3}$$

Since we only use spin-$\frac{1}{2}$ particles and normalized spin projections rather than photons and their polarization states, each time we mention in this paper the singlet state or Malus law (normal or twisted), we mean of course the spin-$\frac{1}{2}$ version of these objects (for a textbook presentation of both of the photons and the spin-$\frac{1}{2}$ particles versions, see for instance [6]).

In the founding paper of Bell's theory ([4], p. 407), Bell reached the conclusion that:

"In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant".

More generally, the structure of a typical Bell type theorem reads either as the following statement that we call *the main implication* or as its consequences as in Bell's citation just above:

Quantum mechanics		*Some inequality is violated*
+ augmentation choice	$\Rightarrow$	*for appropriate choices*
+ extra hypothesis		*of some parameters.*

In the terms of the main implication, the example of "augmentation" chosen in Bell's 1964 paper [4] is the assumption that there are "predictive hidden variables with the same statistics as quantum mechanics" while Bell's original example of "extra hypothesis" is "locality" that we next redefine both more formally and in such a way that the role of the augmentation be clearly stated.

Definition 1. Locality tells us that if (x_0, t_0) and (x_1, t_1) are spatially separated, i.e., $\Delta x^2 > c^2 \Delta t^2$, then the setting of an instrument at (x_0, t_0) cannot change the output of a measurement made at (x_1, t_1). Furthermore, if one assumes that weak realism as defined below holds true, the value of any observable that could be measured at (x_1, t_1) in lieu of the observable that is actually being measured there is also independent of any instrument setting at (x_0, t_0).

In the present paper we will show that the extra hypothesis of Bell's theorem can be chosen to be substantially weaker than locality without affecting the truth of the main implication.

For a long time already many authors have proposed versions of Bell's theorem based on augmentations that are weaker than the predictive hidden variables used in [4]. We recall that the concept of *predictive hidden variables* does not only mean that some variables make sense, even if beyond our reach, but that there are enough such variables so that using all variables, hidden or not, one would get a theory that would not only predict statistical results (like quantum mechanics) but would also predict the result of individual experiments and more generally of all the observables' values (even if one cannot access these values). In particular all usual observables would have well defined values since they would be predictable, so that predictive hidden variables, if they would exist, would imply realism in the sense that observables would have values independently of being observed or not. We shall focus in this paper on an augmentation of quantum mechanics that does not assume any more predictive power than quantum mechanics; more precisely we shall only postulates that there is a value associated to any measurement that could be made on a particle at the time when some measurement is made on that particle. This augmentation, close to but slightly weaker that Stapp's *contrafactual definiteness* [7] (see also [3]), is also implied by the hypothesis used by Bell in [4] and also by what is called sometimes *the EPR condition of reality* [2]. In fact, the augmentation that we choose is constructed as the weakest form of realism sufficient to develop Bell's theory so that we call it simply *weak realism.* Somewhat pushing in the direction opposite to that of Stapp in [7], Lev Vaidman raised the question of whether there is any room between our weak realism and Bell's predictive hidden variables, but the main statements and arguments provided here would not be affected by a negative answer to that question. In fact the readers may choose the formulations of realism at the microscopic level that they like most as the definition of weak realism, in lieu of our minimalist concept. What is most important here is that all the results that we prove by assuming only what we call weak realism could as well be proved by using any (other) form of realistic assumption that can be used to develop Bell's theory. This is because weak realism is nothing but the minimal augmentation of quantum mechanics that assumes that there are as many sequences of values as are needed to develop Bell's theory while preserving the statistical predictions of quantum mechanics.

The concept that we introduce next is another essential ingredient of our work: it will be our extra hypothesis in the main implication, but we will mostly use it in a form that is much more specific that the one proposed here. We have judged that it was preferable to stay at this less technical level in the introduction, and we will also use the following high level description anyhow.

Effect after cause principle (EACP - general form)

(i) For any Lorentz observer the value of an observable cannot change as a result of any cause that happens after said observable has been measured for that observer.
(ii) Furthermore, if one assumes that weak realism holds true, the value of any observable that could be measured at (x, t) where some other observable is measured, but that is only inferred to exist at (x, t) by invoking weak realism, cannot change as a result of any cause that happens after said non-observed observable gets a value at (x, t) as a result of weak realism for that observer.

We notice that time ordering makes sense in the definition of the EACP since it is relative to the chosen Lorentz observer. After showing that the EACP is an hypothesis strictly weaker than locality we will prove a version of Bell's theorem where we only assume weak realism and the EACP. Let us recall that a typical formal statement of Bell's theorem consists in the falsification of some inequality (meaning as usual the exhibition of an instance such that the inequality that one attempts to falsify indeed reduces to a false inequality between two numbers): such a falsification then lets one draw a conclusion as in Bell's citation reported above. Not so surprisingly, there is a price to pay for the increased generality of the Bell's theorem that we will prove in this paper. More specifically, in order to compensate for our weaker assumptions (that consist as usual in some choice of augmentation and an extra hypothesis) the selection of an inequality and of its parameterization needs to be much more controlled in order to produce a falsification of at least one of the Bell's inequalities than what one needs using locality. In particular, avoiding to assume locality will not permit us to falsify any inequality that uses four angles and more precisely two angles for each of the two particles of the singlet state as in the CHSH version of Bell's theorem (see [8–11]). However the original configuration with three angles used in Bell's paper [4] can be dealt with, but then only for some angle on the side where one uses two oriented axes, when one only assumes the EACP. It might be the case that other configurations also work besides the one that we could find, but the fact that we cannot conclude using two angles for each of the two particles of the singlet state will turn out to be deeply linked to the difference between locality and the EACP.

We will also show that weak realism and the EACP together permit to reach a contradiction in the three particles version of the GHZ theorem [12–14]. In that case we can use exactly the same configuration as the one that is used when assuming locality. Otherwise speaking, in the case of the GHZ theorem, and at least in the tree particles version of this result, we do not have to make restrictions when replacing locality by the EACP as what needs to be done to get a Bell's theorem. Recalling that the GHZ theorem is also known as " Bell's theorem without inequality", we can conclude from our results that the only common cause to all contradictions in Bell type theorems is whatever form of realism that one uses to augment quantum mechanics. In particular, invoking non-locality cannot prevent the contradictions that we establish. Theorems of the type of Bell's can thus be used to strongly suggest that weak realism is false (see also [15,16] and Rem. 3 below). Such a conclusion would be in line with the opinion that any form of realism at the microscopic level is in contradiction with the spirit of the uncertainty principle. Rejecting as usual "local realism", i.e., the conjunction of some form of realism and locality, appears to be a misleading conclusion in view of our results.

This paper is organized as follows. In Section 2 we complete the description, started above in this section, of the setting of the *gedanken experiments* dealt with in Bell's theory: for the sake of completeness, simple classical derivations of Bell's inequality and Bell's theorem, in the original and CHSH form, are provided there assuming both weak realism and locality as in the classical Bell's theory. In Section 3, we prove that the EACP is weaker an hypothesis than locality: two of the proofs of that statement will consist in the exhibiting differences between the lists of the correlations that can be computed assuming either locality or the EACP in two versions of Bell's inequalities. We will also provide in Section 3 a computation of one special example of correlation that we call the no correlation lemma. The proof of our version of Bell's theorem is then developed to conclude Section 3. In Section 4 we recall the usual theory of the GHZ theorem in the version based on a three particles maximal entanglement that was proposed by Mermin. Then we will show that for this entanglement as for the singlet state, under the usual assumption of weak realism but then assuming the EACP instead of assuming locality as in prior deductions of the GSZ theorem, one still reaches a contradiction. Since the EACP is weaker than locality, the GHZ entanglement lets one reach yet another contradiction that cannot be removed by dropping the locality assumption. This is one more strong indication that weak realism by itself is false in microphysics with no need to specify that the chosen form of realism needs to be local in order to lead to a contradiction with what we know from quantum mechanics.

2 Setting, statement and proofs in usual Bell's theory

We started the description of the experiment in the introduction, but what was described there of the EPR-Bohm *gedanken experiment* is not yet enough to reach any form of Bell's theorem. The uncertainty principle [17] tells us that only one spin projection, i.e., one axis, can be chosen for each of the two particles p_A and p_B, not enough to generate any meaningful inequality relating different correlations. In order to get enough observables to build a meaningful inequality, one needs to "augment" quantum mechanics into a candidate for a theory of microphysics that would coincide with quantum mechanics where quantum mechanics has something to tell us, and that is compatible with the statistical predictions of quantum mechanics (which have been proven right by numerous experiments over the years). Using the augmentation of quantum mechanics by any form of realism to have more values of observables at once necessarily turns what we started to describe as an experiment into a *gedanken experiment*. Of course, the legitimacy of such an augmentation of quantum mechanics is questionable and we hope that we help to make the case (see Rem. 3) that indeed, weak realism violates the laws of Physics. But this will not prevent us from assuming often weak realism as we argue ad absurdum.

The following two conventions are adopted in a more or less explicit form in all works on Bell's theory, independently of the strength of the augmentation being chosen:

Convention 1. Whenever we assume that quantum mechanics is augmented by a form of realism, we implicitly postulate that any quantity that is not measured but that exists according to the augmentation has the value that would have been measured if this quantity would have been the one measured, the world being otherwise unchanged. It seems to us that the meaning of the value of an observable makes no much sense otherwise so that this convention is probably the most consensual component of this paper.

Convention 2. Whenever we assume that quantum mechanics is augmented by a form of realism, we assume that said augmentation is made without changing the statistical predictions. This is (up to wording) the assumption that Bell made in his foundational 1964 paper [4], except for the fact that we do not restrict the choice of augmentation to predictive hidden variables.

As we shall see, Convention 2 is not enough to get convergence, nor even evaluation for averages over finite sums for all the correlations that we need. Historically, versions of inequalities involving either three sequences of spin projections (what we call *"version $V3$"*) or four sequences of spin projections (what we call *"version $V4$"*) have been used, and it will be important for us to use both versions. More precisely, we will use:

- Version $V3$ in order to get our Bell theorem without locality in Section 3.4.
- Versions $V3$ and $V4$ in order to get two proofs that the EACP is a weaker hypothesis than locality in Section 3.2.

We will also use both versions $V3$ and $V4$ to examine closely in Section 3.2 what would be the cost of abandoning locality when dealing with the usual Bell's theory that uses locality as an essential assumption.

Coming back to the setting and notations of the introduction, and assuming weak realism so that extra axes $(a_A)'_i$ and $(a_B)'_i$ can respectively be chosen by Alice and Bob, we end up having at our disposal the following sequences:

- The sequences $\mathcal{E}_i$ on Alice's side using axes $(a_A)_i$ and $\mathcal{P}_i$ on Bob's side using axes $(a_B)_i$ are the two sequences of normalized spin projections that are actually observed.
- The sequences $\mathcal{E}'_i$ on Alice's side and $\mathcal{P}'_i$ on Bob's side that are the two sequences of what would be supplementary normalized spin projections, with values that are most probably out of reach. Such supplementary normalized spin projections values would be well defined – even if out of possible knowledge – if and only if weak realism or some stronger form of microscopic realism holds true (hence the statement that weak realism is the weakest form of realism that can be used to develop Bell's theory). These sequences are supposedly what one would get respectively along the axes $(a_A)'_i$ and $(a_B)'_i$ if those axes would be used to measure normalized spin projections instead of the axes $(a_A)_i$ and $(a_B)_i$. Even if such supplementary sequences of normalized spin projections cannot be known, one may construct out of them some objects with statistical significance such as correlations or probabilities of equality on which one has grip under the standing assumption that whichever form of microscopic realism that one invokes must respect the statistical predictions of quantum mechanics.

One may think that the range of the index i is cut into disjoint intervals I_κ so that for any I_κ the axes $(a_A)_i$, $(a_B)_i$, $(a_A)'_i$, $(a_B)'_i$ do not vary with i as long as i stays in I_κ: we shall denote by N_κ the number of elements of I_κ. All the sequences that we have introduced are sequences of normalized spin projections for spin-$\frac{1}{2}$ particles, hence sequences of -1's and 1's. We shall next focus on abstract sequences and finite chunks of sequences of 1's and -1's.

2.1 The formal aspects of Bell's inequalities

We now follow Sica [18,19] (except that we defer deciding which quantities need a prime) who noticed that if w_i, x_i, y_i and z_i are four sequences with values in the set $\{-1, 1\}$, then one has simple factorization identities that lead via simple algebra to inequalities involving either three or four sequences or finite chunks of these sequences. For version $V3$, we use that $y_i^2 \equiv 1$ to start with:

$$x_i y_i - x_i z_i = x_i y_i \left(1 - y_i z_i\right) \tag{4}$$

so that by summing over the elements of I_κ, dividing by N_κ and taking absolute values, we get:

$$\left| \sum_{i \in I_\kappa} \frac{x_i y_i}{N_\kappa} - \sum_{i \in I_\kappa} \frac{x_i z_i}{N_\kappa} \right| \leq \sum_{i \in I_\kappa} \frac{|x_i y_i| \, |1 - y_i z_i|}{N_\kappa} \leq 1 - \sum_{i \in I_\kappa} \frac{y_i z_i}{N_\kappa}. \tag{5}$$

Thus

$$\left| \sum_{i \in I_\kappa} \frac{x_i y_i}{N_\kappa} - \sum_{i \in I_\kappa} \frac{x_i z_i}{N_\kappa} \right| \leq 1 - \sum_{i \in I_\kappa} \frac{y_i z_i}{N_\kappa}. \tag{6}$$

Assume then that there is convergence as $N_\kappa \to \infty$. Denoting by $\langle f, g \rangle$ the correlation of two functions f and g, we get:

$$|\langle x, y \rangle - \langle x, z \rangle| \leq 1 - \langle y, z \rangle, \tag{7}$$

one formal form of the $V3$ version of Bell's inequalities.

We now turn to the algebra of the version $V4$. Again following Sica we start with:

$$x_i y_i + x_i z_i + w_i y_i - w_i z_i = x_i\,(y_i + z_i) + w_i\,(y_i - z_i)\,. \quad (8)$$

Simple manipulations on this identity then yield:

$$\left|\frac{1}{N_\kappa}\sum_{I_\kappa} x_i y_i + \frac{1}{N_\kappa}\sum_{I_\kappa} x_i z_i\right| + \left|\frac{1}{N_\kappa}\sum_{I_\kappa} w_i y_i - \frac{1}{N_\kappa}\sum_{I_\kappa} w_i z_i\right| \le \frac{1}{N_\kappa}\sum_{I_\kappa} |x_i|\,|y_i + z_i| + \frac{1}{N_\kappa}\sum_{I_\kappa} |w_i|\,|y_i - z_i|\,. \quad (9)$$

Now, since $\min(|y_i+z_i|, |y_i-z_i|) = 0$ and $\max(|y_i+z_i|, |y_i-z_i|) = 2$, equation (9) can be rewritten as

$$\left|\frac{1}{N_\kappa}\sum_{I_\kappa} x_i y_i + \frac{1}{N_\kappa}\sum_{I_\kappa} x_i z_i\right| + \left|\frac{1}{N_\kappa}\sum_{I_\kappa} w_i y_i - \frac{1}{N_\kappa}\sum_{I_\kappa} w_i z_i\right| \le 2. \quad (10)$$

Assuming convergence, the averages generate correlations and one obtains the following form of the CHSH inequality, our $V4$ version of Bell's inequalities

$$|\langle x, y\rangle + \langle x, z\rangle| + |\langle w, y\rangle - \langle w, z\rangle| \le 2, \quad (11)$$

which contains equation (7) as a special case (first restrict to $x = y$, replace each x by y and then rename w to x). We notice that when two sequences are actually observed so that elements with the same index come from the same pair, then quantum mechanics provides the value of the correlation and in particular guarantees convergence. Be it in version $V3$ or version $V4$, we made no attempt to deduce all the Bell's inequalities, formal or not. For that and the statistical aspects of Bell's inequalities and Bell's inequalities as a particular case of Boole's inequalities, see for instance [20–26].

2.2 From formal inequalities to Bell's inequalities and Bell's theorem

We have obtained versions $V3$ and $V4$ of Bell's inequalities using abstract sequences of 1's and -1's. In order to come one step closer to physics, we first appropriately pair:

- the symbols $\mathcal{E}_i$, $\mathcal{P}_i$ that represent actual observations,
- and the symbols $\mathcal{E}'_i$ and $\mathcal{P}'_i$ that represent values provided by the weak realism assumption to the sequences w_i, x_i, y_i, z_i used in deriving the inequalities (6) and (10).

For version $V3$, we need to take x_i and z_i on the same side, e.g., Alice's side: thus $x_i = \mathcal{E}_i$ and $z_i = \mathcal{E}'_i$, whence $y_i = \mathcal{P}_i$. Then equations (6) and (7) become respectively:

$$\left|\sum_{i\in I_\kappa}\frac{\mathcal{E}_i\mathcal{P}_i}{N_\kappa} - \sum_{i\in I_\kappa}\frac{\mathcal{E}_i\mathcal{E}'_i}{N_\kappa}\right| \le 1 - \sum_{i\in I_\kappa}\frac{\mathcal{P}_i\mathcal{E}'_i}{N_\kappa} \quad (12)$$

and

$$|\langle\mathcal{E}, \mathcal{P}\rangle - \langle\mathcal{E}, \mathcal{E}'\rangle| \le 1 - \langle\mathcal{P}, \mathcal{E}'\rangle\,. \quad (13)$$

As for version $V4$, we want w and z to be the values generated by the weak realism hypothesis, but we need also x_i and z_i to be on different sides and y_i and w_i to be on different sides. One way to achieve that is to choose the replacements $x \to \mathcal{E}$, $y \to \mathcal{P}$, $w \to \mathcal{E}'$, $z \to \mathcal{P}'$. Thus equations (10) and (11) become respectively:

$$\left|\frac{1}{N_\kappa}\sum_{I_\kappa}\mathcal{E}_i\mathcal{P}_i + \frac{1}{N_\kappa}\sum_{I_\kappa}\mathcal{E}_i\mathcal{P}'_i\right| + \left|\frac{1}{N_\kappa}\sum_{I_\kappa}\mathcal{E}'_i\mathcal{P}_i - \frac{1}{N_\kappa}\sum_{I_\kappa}\mathcal{E}'_i\mathcal{P}'_i\right| \le 2 \quad (14)$$

and the following form of the CHSH inequality:

$$|\langle\mathcal{E}, \mathcal{P}\rangle + \langle\mathcal{E}, \mathcal{P}'\rangle| + |\langle\mathcal{E}', \mathcal{P}\rangle - \langle\mathcal{E}', \mathcal{P}'\rangle| \le 2. \quad (15)$$

Our main goal in this section is only to reach the classical Bell's inequalities and Bell's theorems under the usual hypothesis. We also want to inspect here the correlations that can be computed if one assumes weak realism and locality. This examination of what is computable under these hypotheses will be used in the next section to compare the strengths of different hypotheses.

We have already invoked weak realism in order to give meaning to three spin projections at once in version $V3$, or four spin projections at once in version $V4$. In order to give meaning to the correlations in equations (13) and (15), we now further assume locality, so that the sequences on one side do not depend on the choice of the axes along which the spin is projected on the other side. Then, under Conventions 1 and 2 that are both triggered by assuming weak realism, we can use the twisted Malus law, that gives us:

$$\langle\mathcal{E}, \mathcal{P}\rangle = -\cos\left(\theta_{\mathcal{E}} - \theta_{\mathcal{P}}\right), \quad (16)$$

to also obtain readily:

$$\langle\mathcal{E}, \mathcal{P}'\rangle = -\cos\left(\theta_{\mathcal{E}} - \theta_{\mathcal{P}'}\right) \quad (17)$$

and

$$\langle\mathcal{E}', \mathcal{P}\rangle = -\cos\left(\theta_{\mathcal{E}'} - \theta_{\mathcal{P}}\right). \quad (18)$$

Let $\tilde{\mathcal{Q}}$ stand for the sequence or normalized spin projections along the angle $\theta_{\mathcal{Q}}$ but on the side opposite to the side corresponding to $\mathcal{Q}$. Since in this section we are assuming locality, we have the identity:

$$\tilde{\mathcal{Q}}_i + \mathcal{Q}_i \equiv 0 \tag{19}$$

for any $\mathcal{Q} \in \{\mathcal{E}, \mathcal{P}, \mathcal{E}', \mathcal{P}'\}$. The relation (19) is a direct consequence of the singlet state expression and wave packet reduction if one at least of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ is actually measured. For the other cases, one uses locality to state that $\tilde{\mathcal{Q}}_i$ is unchanged if the setting is changed on the other side, where one could actually measure $\mathcal{Q}$. But then, we notice that by Convention 1 and locality, $\mathcal{Q}_i$ remains unchanged if it is measured instead of being inferred to make sense by invoking weak realism, so that in all cases the conclusion is the same as if one at least of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ is observed.

We notice that if one does not assume locality, then the identity (19) holds true when at least one of $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ is actually observed, but not necessarily otherwise since one cannot then use the reasoning on which we relied to justify the relation (19) in the case when locality is assumed to hold true. This difference between what can be deduced depending on whether one assumes or not locality to hold true will be very important in the next section.

From equations (17) or (18) that are equivalent to each other by exchanging the sides of Alice and Bob, we get readily:

$$\left\langle \mathcal{E}, \tilde{\mathcal{E}}' \right\rangle = -\cos\left(\theta_{\mathcal{E}} - \theta_{\mathcal{E}'}\right) \tag{20}$$

and

$$\left\langle \tilde{\mathcal{P}}', \mathcal{P} \right\rangle = -\cos\left(\theta_{\mathcal{P}'} - \theta_{\mathcal{P}}\right), \tag{21}$$

from which by (19) we respectively get:

$$\langle \mathcal{E}, \mathcal{E}' \rangle = \cos\left(\theta_{\mathcal{E}} - \theta_{\mathcal{E}'}\right) \tag{22}$$

and

$$\langle \mathcal{P}', \mathcal{P} \rangle = \cos\left(\theta_{\mathcal{P}'} - \theta_{\mathcal{P}}\right). \tag{23}$$

Using again (22), (23), and locality, we also know that:

$$\left\langle \mathcal{E}', \tilde{\mathcal{P}}' \right\rangle = \cos\left(\theta_{\mathcal{E}'} - \theta_{\mathcal{P}'}\right) \tag{24}$$

and

$$\left\langle \mathcal{P}', \tilde{\mathcal{E}}' \right\rangle = \cos\left(\theta_{\mathcal{P}'} - \theta_{\mathcal{E}'}\right). \tag{25}$$

Any of these two equations lets us compute $\langle \mathcal{P}', \mathcal{E}' \rangle$ as:

$$\langle \mathcal{E}', \mathcal{P}' \rangle = -\cos\left(\theta_{\mathcal{E}'} - \theta_{\mathcal{P}'}\right). \tag{26}$$

We now have the values, hence also in particular the convergence of the finite sums as the numbers N_κ diverge, for all the correlations that we need in both versions $V3$ and $V4$. Thus both of the Bell's inequalities, i.e., equations (13) and (15), that we have formally deduced assuming convergence are fully justified if one assumes weak realism and locality. In order to get from Bell's inequalities to Bell's theorem one needs to falsify at least one of these inequalities by choosing appropriate values of the parameters (the oriented axes or equivalently the angles). We will provide falsifications for both versions $V3$ and $V4$.

- For version $V3$ we choose $\theta_{\mathcal{P}} = 0$, $\theta_{\mathcal{E}} = \frac{3\pi}{4}$, and $\theta_{\mathcal{E}'} = \frac{-3\pi}{4}$ so that $\theta_{\mathcal{E}}$ and $\theta_{\mathcal{E}'}$ differ by a right angle. Since using locality we easily get $\langle \mathcal{E}, \mathcal{E}' \rangle = 0$, by further using $\langle \mathcal{E}, \mathcal{P} \rangle = \langle \mathcal{E}', \mathcal{P} \rangle = \frac{\sqrt{2}}{2}$, and by replacing all the correlations in equation (13) by their respective values we end up deducing the false inequality $\sqrt{2} < 1$ by specialization of the $V3$ version of Bell's inequalities. We can thus conclude that at least one of the assumptions that we have made, weak realism and locality, must be a violation of the laws of microphysics. Many other choices of angles would also work to generate a falsification of equation (13) or another Bell's inequality. **Q.E.D.**
- For version $V4$ we choose $\theta_{\mathcal{E}} = \frac{\pi}{4}$, $\theta_{\mathcal{E}'} = \frac{3\pi}{4}$, $\theta_{\mathcal{P}} = \frac{\pi}{2}$, and $\theta_{\mathcal{P}'} = 0$. thus angular differences $|\theta_{\mathcal{E}} - \theta_{\mathcal{P}}| = |\theta_{\mathcal{E}} - \theta_{\mathcal{P}'}| = |\theta_{\mathcal{E}'} - \theta_{\mathcal{P}}| = \frac{\pi}{4}$ and $|\theta_{\mathcal{E}'} - \theta_{\mathcal{P}'}| = \frac{3\pi}{4}$. Replacing the correlations in equation (15) by their respective values we end up having deduced the false inequality $2\sqrt{2} \leq 2$ by specialization of the $V4$ version of Bell's inequalities. Thus we can again conclude that at least one of the assumptions that we have made, weak realism and locality, must be a violations of the laws of microphysics. Many other choices of angles would also work here, but the example chosen here for $V4$ is optimal in terms of the worse falsification of (15). **Q.E.D.**

Remark 1. As Sica noticed in [18], the finite N_κ equations (12) and (14) are identities, independently of any convergence property. Sica calls them "Bell identities" to distinguish them form the "Bell inequalities" that follow from these identities if convergence hold true for all the averages. The Bell identities have to be satisfied as soon as one assumes weak realism that provides us the three or four sequences of -1's and 1 that are needed depending on which of these two identities we want to work with. Nevertheless, it would at best hard to use the Bell identities to get a contradiction if one had no proof of convergence since then one would not have means to evaluate the terms in the identities (whether one deals with finite sums or with their asymptotic values).

3 A Bell's theorem with no locality assumption

In Section 2.2, we have recalled the classical theory of Bell, in two versions $V3$ and $V4$ where the number in the name of the version is the number of oriented axes used to obtain normalized projections of the spins. We completed this task assuming weak realism and locality.

But what happens if locality is replaced by the EACP?

We will first investigate what remains of the computability of the various correlations that are related by one or another Bell inequality. We will see that only $V3$ can be dealt with when assuming the EACP instead of locality: one of the correlations in equation (15) cannot be evaluated, nor even guaranteed convergence with the substitute hypothesis.

3.1 Statement of the EACP vs. locality lemma

We will use, here and in the next section, a version of the effect after cause principle that is quite focused on the entities that we deal with in Bell's theories.

Effect after cause principle (EACP). For any Lorentz observer and for any $\mathcal{Q}$ in $\{\mathcal{E}, \mathcal{E}', \mathcal{P}, \mathcal{P}'\}$, a value $\mathcal{Q}_i$ of $\mathcal{Q}$ cannot change as a result of a cause that happens after $\mathcal{Q}_i$ has been measured for that observer.

This version of the EACP adapted to the context of Bell's theory will be used to prove the following result that is crucial to our purpose:

EACP vs. locality lemma. The EACP is different from locality, and in fact strictly weaker than locality.

Otherwise speaking, locality implies the EACP but the reverse implication is not true.

Before proving this lemma, we give a definition that will be useful whenever dealing with the EACP throughout the rest of the paper.

Definition 2. With $(X, Y) \in \{(E, P), (P, E)\}$ an X-Y observer is a Lorentz observer for whom measurements at the measurement tool X occur before measurements at measurement tool Y for each pair produced at S.

3.2 Proofs of the EACP vs. locality lemma

First proof of the EACP vs. locality lemma. We assume weak realism and the EACP and notice that in the setting that corresponds to the version $V3$ of Bell's inequality, only one angle is used on one of the sides of Alice and Bob, say on Bob's side with no loss of generality. As a consequence the E-side sequences may depend on $\mathcal{P}_i$ for P-E observers. Furthermore, our E-P observers will not even (need to) consider what happens on the P side in order to analyze the aspects of the E side that we will use. The P-E observers can tell us that:

- The correlation $\langle \mathcal{P}, \mathcal{E} \rangle$ is well defined and has the right minus cosine value determined by quantum mechanics.
- The correlation $\langle \mathcal{P}, \mathcal{E}' \rangle$ is well defined and has the right minus cosine value by a simple application of the weak realism hypothesis and the EACP. This can be compared to the previous deduction of formula (17) from the assumption of locality: notice that we do not assume locality here, but only the EACP instead.

In both of the cases of $\langle \mathcal{P}, \mathcal{E} \rangle$ and $\langle \mathcal{P}, \mathcal{E}' \rangle$, the correlation is indeed given by the twisted Malus law. As for the status of $\langle \mathcal{E}, \mathcal{E}' \rangle$ (or of $\langle \mathcal{P}, \mathcal{P}' \rangle$ if we decide that the choice is on Bob's side), for general mutual positions of the axes, the conjunction of quantum mechanics, weak realism, and the EACP does not provide us with any mean to evaluate these correlations: this incapacity may be hard to accept as proving anything, which is why we will also provide a third, quite different, method to prove the EACP vs. locality lemma. However by using symmetry considerations we are going to evaluate them when $a_{\mathcal{E}}$ is orthogonal to $a_{\mathcal{E}'}$ in the next section; this will be the object of the *no-correlation* lemma. Of course, one could take locality as an extra specification of the EACP and get back the values of correlations computed above in Section 2.2, but this would mean using an extra hypothesis. Thus, the first element of comparison between the EACP and locality is to notice that indeed, the EACP by itself cannot lead to a computation of $\langle \mathcal{E}, \mathcal{E}' \rangle$ nor of $\langle \mathcal{P}, \mathcal{P}' \rangle$ for general angles $\langle a_{\mathcal{E}}; a_{\mathcal{E}'} \rangle$ or $\langle a_{\mathcal{P}}; a_{\mathcal{P}'} \rangle$. **Q.E.D.**

Second proof of the EACP vs. locality lemma. Turning now to the case of version $V4$, assuming weak realism and the EACP a P-E observer will be able to recognize that $\langle \mathcal{P}, \mathcal{E}' \rangle = -\cos(\theta_{\mathcal{P}} - \theta_{\mathcal{E}'})$. Indeed the EACP guarantees that, for such a Lorentz observer, the sequence $\{\mathcal{P}_i\}$ cannot change if Alice chooses $\theta_{\mathcal{E}'}$ instead of $\theta_{\mathcal{E}}$. Similarly, exchanging the roles of the two sides, an E-P observer will be able to recognize that $\langle \mathcal{E}, \mathcal{P}' \rangle = -\cos(\theta_{\mathcal{E}} - \theta_{\mathcal{P}'})$. We also have of course $\langle \mathcal{E}, \mathcal{P} \rangle = -\cos(\theta_{\mathcal{E}} - \theta_{\mathcal{P}})$.

This is not enough however to let one use the CHSH inequality with four different oriented axes and in particular without specializing version $V4$ to version $V3$: what is missing is the mean to evaluate $\langle \mathcal{E}', \mathcal{P}' \rangle$ under the EACP assumption. With the EACP replacing locality, no consequence of quantum mechanics augmented by weak realism can be used to help us compute $\langle \mathcal{E}', \mathcal{P}' \rangle$ or even to only guaranty the convergence of $\frac{1}{N_\kappa} \sum_{i \in I_\kappa} \mathcal{E}'_i \mathcal{P}'_i$ as $N_\kappa \to \infty$. The case when $\theta_{\mathcal{E}'} \equiv \theta_{\mathcal{P}'} \bmod \pi$ could be erroneously considered as special in terms of computability of $\langle \mathcal{E}', \mathcal{P}' \rangle$: at first inspection, it would seem that one can then compute $\langle \mathcal{E}', \mathcal{P}' \rangle$ by using the conservation law embedded in the singlet state. However a close inspection reveals that any such computation would in fact use locality, and the same would apply if one would compute $\langle \mathcal{Q}, \mathcal{Q}' \rangle$ as $\langle \mathcal{Q}, \tilde{\mathcal{Q}}' \rangle$ for $Q \in \{E, P\}$. For people unconvinced that our incapacity to compute $\langle \mathcal{E}, \mathcal{E}' \rangle$ or $\langle \mathcal{P}, \mathcal{P}' \rangle$ under the EACP assumption while the computation of these quantities can be done if we assume locality, we point out our incapacity to compute $\langle \mathcal{E}', \mathcal{P}' \rangle$ under the EACP assumption is accompanied by a problem of a quite different nature. Indeed, assuming only the EACP, an important information about the sequence $\mathcal{E}'$ and $\mathcal{P}'$ is missing when the values at the corresponding angles are computed together: assuming the EACP does not comprise assuming locality so that the primary non local effect of dependence on the setting of the measurement tool on the other side is not excluded. As a consequence, even the evaluation of $\langle \mathcal{E}', \mathcal{P}' \rangle$ would not let one write a legitimate Bell inequality. As Boole already knew in 1862 [22], the data with any given index have to come from the same experiment, and $\mathcal{E}'$ and $\mathcal{P}'$ need a priori another instance if one assumes the EACP instead of locality, even if only a *gedanken* instance, to make sense as a pair of compatible values, thus preventing the application of a Boole-Bell inequality.

As mentioned in the case of version $V3$, one could take locality as an extra specification of the EACP and get back the values of correlations computed in Section 2.2, but like in the case of version $V3$ this would mean using an extra hypothesis. Thus, the second element of comparison between the EACP and locality is to notice that indeed, the

EACP by itself cannot lead to a computation of $\langle \mathcal{E}', \mathcal{P}' \rangle$ nor to a guaranty that the $\mathcal{E}'$ (respectively the $\mathcal{P}'$) is the same when $\mathcal{P}$ is measured and when $\mathcal{P}'$ is measured (respectively the same when $\mathcal{E}$ is measured and when $\mathcal{E}'$ is measured), so that even computing $\langle \mathcal{E}', \mathcal{P}' \rangle$ would not let us write a legitimate inequality of the type discussed by Boole and Bell. As a consequence, there is no $V4$ version of Bell's theorem when assuming only the EACP instead of locality, and in particular there is no CHSH version of Bell's theorem when one uses the EACP instead of locality. This constitutes another difference between the EACP and locality. **Q.E.D.**

Third proof of the EACP vs. locality lemma. We aim here at the same lemma but for the EACP as it appears in its general definition given in the introduction. For that, we first notice that the EACP is nothing but causality in a world without augmentation of quantum mechanics by any form of realism. On the other hand causality is known to be the impossibility of signaling. Since one knows that the violation of locality, independently of realism, does not necessarily permit signaling (see, e.g., [27]), we know that locality and causality do not coincide in a world without augmentation of quantum mechanics by any form of realism: indeed locality is stronger that causality in such a world. We deduce that the EACP is not locality, and is in fact weaker than locality in a world without augmentation of quantum mechanics by any form of realism.

While we aim at proving that we are indeed in a world without augmentation of quantum mechanics by any form of realism, we cannot use this fact and need to make sure of what happens if one assumes that weak realism holds true. Then the negation of the EACP for non-observed observables is in conflict with Convention 1 so that, like in the case when one does not assume that weak realism holds true, the negation of the EACP can only have a chance to hold true if causality fails. Thus, even in a world where weak realism holds true, the EACP is not stronger than causality, and is thus weaker than locality. **Q.E.D.**

Details are left to the reader but one reaches here the fine line where Physics turns into Metaphysics: by negating the EACP, one gets to handle entities that, by nature, escape any experimentation (see also Rem. 3 below). This is why we have also provided the two down to Earth proofs of the EACP vs. locality lemma. However, the first proof presents a different problem serious enough to call for a third, quite different proof: we expect that our failure to find means to evaluate some correlations in the first and second proofs of the EACP vs. locality lemma will be accepted as a serious absence of means to compute these correlations, and in the second proof, we have a more trustable difference since the EACP does *not* tell us that the value of a measurement on one particle of an EPR pair may not depend on the setting of the measurement tool used for the other particle. Having seen now the third proof, we may be content with the convergence of the arguments of all kinds. Nevertheless we notice that in the framework of the first and second proofs, it would be interesting to get a more formal proof of the impossibility to compute, or a more formal proof of the lemma in its very operational form by some other means.

Remark 2. The first part of third proof of the EACP vs. locality lemma tells us that the negation of the EACP in a world without augmentation permits signaling, which is enough to prove the EACP vs. locality lemma in such a world. Unfortunately, this first part does not exactly tell us that the EACP is trivially true as just being causality since the EACP might still fail (only) in the limbo of entities that are well defined only because realism in some form holds true. Since such a world could be the actual world (although the final conclusion will be to contrary), we had to also consider a world accepting weak realism in the third proof. While a formal proof that the EACP is not locality is achieved by restricting to a world with no realism at the microscopic level, this would not prove the EACP vs. locality lemma as long as weak realism has not been proved wrong and we have to cover both cases to not fall in a circular argument. See also Remark 3 in Section 3.3.

3.3 The no-correlation lemma

No correlation lemma. Assuming the EACP, if the oriented axes $a_{\mathcal{E}}$ and $a_{\mathcal{E}'}$ are orthogonal to each other, then the sequences $\mathcal{E}$ and $\mathcal{E}'$ are not correlated, i.e.,

$$(\circ) \quad \langle \mathcal{E}, \mathcal{E}' \rangle = 0 \quad \text{or equivalently} \quad \mathrm{Prob}\,(\mathcal{E}_i = \mathcal{E}'_i) = \frac{1}{2}.$$

Proof of the no correlation lemma. Using the EACP and the conclusion deduced from it, and more precisely the fact that "the three sequences $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{P}$ involved in equation (13) are well defined", we notice that only the orientation of the angle $\langle a_{\mathcal{E}}; a_{\mathcal{E}'} \rangle$ at E could matter for an E-P observer, so that $(\circ)$ follows from invariance under parity without assuming locality. We next provide some details that some readers may prefer to avoid.

To see the role of parity, we introduce the further oriented axis $a_{\mathcal{E}''}$ that is parallel to $a_{\mathcal{E}'}$ but with the opposite orientation. This is the (only) oriented axis to which would correspond the sequence $\mathcal{E}''$ such that $\mathcal{E}''_i \equiv 1 - \mathcal{E}'_i$. Since

$$\mathrm{Prob}\,(\mathcal{E}_i = \mathcal{E}'_i) + \mathrm{Prob}\,(\mathcal{E}_i = \mathcal{E}''_i) = 1, \qquad (27)$$

it only remains to prove that these two probabilities are equal to each other. We use here sequences whose values are possibly unknown (and indeed forever inaccessible to our knowledge), but that are known to be well defined as we have recalled to begin this proof:

- one of these sequences, $\mathcal{E}$, is known by direct measurement,
- the other sequence, $\mathcal{E}'$, can be inferred to be well defined, even if unknown, by an E-P observer on the basis of quantum mechanics augmented by weak realism.

Since the angles $\langle a_{\mathcal{E}}; a_{\mathcal{E}'} \rangle$ and $\langle a_{\mathcal{E}''}; a_{\mathcal{E}} \rangle$ are equal, using the EACP, the only thing that could generate an inequality between $\mathrm{Prob}(\mathcal{E}_i = \mathcal{E}'_i)$ and $\mathrm{Prob}(\mathcal{E}_i = \mathcal{E}''_i)$ for an E-P observer is the difference in the orientations of the angles $\langle a_{\mathcal{E}}; a_{\mathcal{E}'} \rangle$ and $\langle a_{\mathcal{E}}; a_{\mathcal{E}''} \rangle$. Equality thus follows from parity invariance. **Q.E.D.**

3.4 A Bell's theorem with no locality assumption

In this section, we formulate and prove the following main result.

New Bell's theorem. Assuming weak realism and the EACP, we can use the triplet of angles $(\theta_{\mathcal{P}}, \theta_{\mathcal{E}}, \theta_{\mathcal{E}'}) = (0, \frac{3\pi}{4}, \frac{-3\pi}{4})$ that corresponds to the triplet of correlations $(\langle \mathcal{P}, \mathcal{E} \rangle, \langle \mathcal{E}', \mathcal{P} \rangle, \langle \mathcal{E}, \mathcal{E}' \rangle) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$ to generate a contradiction using the $V3$ version of Bell's inequalities.

Proof of the new Bell's theorem. Again we assume the EACP and we use equation (13) as the inequality to be falsified.

(1) After measurements are made using P, a P-E observer obtains that:
   (e1) $\langle \mathcal{P}, \mathcal{E} \rangle = \frac{\sqrt{2}}{2}$, i.e., $\langle \mathcal{P}, \mathcal{E} \rangle \approx 0.7$ by quantum mechanics, or by direct observation after measurements are also made using E,
   (e2) $\langle \mathcal{P}, \mathcal{E}' \rangle = \frac{\sqrt{2}}{2}$, i.e., $\langle \mathcal{P}, \mathcal{E}' \rangle \approx 0.7$ by quantum mechanics augmented by weak realism.
   The deductions made in (e1) and (e2) using quantum mechanics augmented by weak realism go as follows: by wave packet reduction (for instance), the spin state of second particle (the particle on the E side) becomes
   $$\Psi\,(x_2) = |\mathcal{P}_i\rangle_1 \otimes | - \mathcal{P}_i\rangle_2 \tag{28}$$
   along the oriented axis along which the sequence $\mathcal{P}$ is measured, as soon as the measurement of $\mathcal{P}_i$ is made on the P side. Hence the second particle gets into a spin state prepared to be $| - \mathcal{P}_i\rangle$ along that oriented axis (as revealed by using the information obtained on the P side) so that both of the two correlations $\langle \mathcal{P}, \mathcal{E} \rangle$ and $\langle \mathcal{P}, \mathcal{E}' \rangle$ are equal to $\frac{\sqrt{2}}{2}$ (about 0.7) by a simple application of the twisted Malus law as we have recalled it, under Convention 2 and the EACP, as we saw in Section 2.2.
(2) An E-P observer infers that:
   (e3) $\mathrm{Prob}(\mathcal{E}_i = \mathcal{E}'_i) = 0.5$ (i.e., $\langle \mathcal{E}, \mathcal{E}' \rangle = 0$) on the E side by the no correlation lemma.

Assembling the conclusions (e1), (e2), and (e3) from the two (strongly) asynchronous frames (e.g., in the Lorentz frame of the experiment since the outcomes cannot change according to the Lorentz frame by relativistic invariance of observable events) one obtains the expected triplet evaluation for the three correlations: $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$. Together with equation (13) (the $V3$ version of Bell's inequalities), this evaluation provides us with the impossible inequality $1.4 \leq 1$, or more precisely $\sqrt{2} \leq 1$ as when we examined equation (13) while assuming locality in Section 2.2. This concludes the proof of the new Bell's theorem. **Q.E.D.**

Our new Bell's theorem admits the following immediate corollary that we will use as our main conclusion:

Conclusive corollary. Weak realism is the **only** possible cause of contradiction common to **all** the versions of Bell's theorem and some of the Bell's type contradictions cannot be solved by assuming non-locality. Thus non-locality is not needed (in some circles, one would say that non-locality can be disposed of using Occam's razor).

Remark 3. As we saw, without (weak) realism any violation of the EACP is a violation of causality since then the EACP is one of the expressions of causality. In order for the violation of the EACP to not be a violation of causality, one would have to accept that, with probability one, the negation of the EACP has effect only on values of observable that are linked to (weak) realism. We do find that unacceptable and we thus consider that the new Bell theorem condemns weak realism. This is not a proof and possibly no actual proof can be given to help us decide between keeping weak realism and keeping the EACP but this situation is more frequent than one might think. Mathematics and not physics is the realm of "proofs": there has always been some opinions lurking behind the way we apprehend the laws of Physics. Some would say that as soon as one uses weak realism, one steps into Metaphysics anyway. However we point out that weak realism violates the spirit if not the letter of the uncertainty principle [17] or at least its time-reversed version [15,16], and is in particular rejected by the Copenhagen interpretation of quantum mechanics (which consequently would have to statute that Bell's theorem has nothing to say about quantum mechanics). On the other hand, invoking weak realism, even if bad, is probably not as bad as accepting that the EACP is false, yet the present work shows that accepting the mildly unacceptable weak realism implies accepting the quite unacceptable violation of the EACP.

Remark 4. The well known hidden variable theory of de Broglie and Bohm [28,29] is both non-local and realist, yet it avoids the contradictions that form Bell's theorem. In fact it avoids these contradictions precisely because non-locality prevents the Bell's inequality from making sense. This statement on the de Broglie-Bohm theory (dBBT) is not a contradiction to our conclusive corollary about weak realism and locality nor more generally to the theses defended here. Indeed, the dBBT is massively not Lorentz invariant, way beyond the special setting for Bell's theory, and apparently irreducibly so: in any Bohmian quantum theory the quantum equilibrium distribution $|\Psi|^2$ cannot simultaneously be realized in all Lorentz frames of reference. To the contrary, quantum mechanics can be viewed as a non-relativistic approximation to relativistic quantum field theory in the limit when classical mechanics is a good approximation to special relativity. The dBBT, or Bohmian mechanics, the version re-discovered and extended by Bohm, is thus false or at least considered as false by most physicists, even if it can serve pedagogically as advocated by Bell (this opinion of Bell, who defended Bohmian mechanics, may not be shared by those who consider non-realism as an essential ingredient of microphysics). Indeed, some Bohmian physicists still hope for a version of Bohmian mechanics that could be acceptable by the profession, but it should be noted that (many) Bohmian physicists take as a strong argument in favor of Bohmian mechanics the false fact that Bell's theorem and Aspect's experiments prove quantum mechanics to be a non-local theory. Indeed, in some sense, Bell's theorem can be considered as the proof that the non-local character of the dBBT was irreducible. To the contrary since it assumes (weak or stronger) realism, hence forces us out of quantum mechanics (into Bohmian mechanics or some weak form of it), Bell's theory and related experiments have nothing to say about the locality or non-locality of quantum mechanics itself, only statements about some augmentations of quantum mechanics.

4 The GHZ theorem, from locality to the EACP hypothesis

Our goal in this section is to extend the class of Bell theorems to which the conclusive corollary stated in the previous section would apply. For that we use a special, three particles, version due to Mermin [13] (see also [14]) of the *GHZ theorem*, a result also known as *"the Bell theorem with no inequality"*, and where GHZ stands for Greenberger, Horne, and Zeilinger, the author of the original, four particles, version of this result [12].

4.1 The (gedanken) experiments of the GHZ type

The GHZ setting is a type of entanglement different from what one studies in the EPR context. Initially conceived with 4 particles [12], such entanglements were built to show that local realism was indeed more obviously wrong (in the sense of "without calling upon statistics", or formally "without inequalities") than what could be established by using a falsification of Bell's inequalities [4]. Further entanglements of the GHZ type were later conceived with 3 particles [13,14]. We shall use here the version in [13] that has made it to the lab and to textbooks (see, e.g., [5] or [6]). Restricting to the spin part, the state that one considers reads:

$$\Psi\,(x_1,x_2,x_3) = \frac{1}{\sqrt{2}}\,(\mid +\rangle_1 \otimes \mid +\rangle_2 \otimes \mid +\rangle_3 - \mid -\rangle_1 \otimes \mid -\rangle_2 \otimes \mid -\rangle_3)\,. \tag{29}$$

The three particles travel out from near $(0,0,0)$ in the plane $y=0$, the particle labelled $k \in \{1,2,3\}$ approximately along the half-line starting at the origin and making an angle $2k\pi/3$ with some reference half-line in the plane $y=0$. As we shall see in in the next section, assuming both weak realism and locality leads to a contradiction.

4.2 The contradiction attached to the GHZ entanglement

The three particles travel out from near $(0,0,0)$ in the plane $y=0$, the particle labelled $k \in \{1,2,3\}$ approximately along the half-line starting at the origin and making an angle $2k\pi/3$ with some reference half-line in the plane $y=0$. We now assume both weak realism and locality. For each $k \in \{1,2,3\}$, and $w \in \{x,y,z\}$, $\Sigma_w(k)$ is the spin operator along the axis $w(k)$, where:

- $y(k) \equiv y$ is the vertical axis, oriented positively upward (this axis is of course independent of k),
- $z(k)$ is the axis along which particle k travels,
- $x(k)$ is orthogonal to y and $z(k)$ and oriented positively counterclockwise.

We have for any $k \in \{1,2,3\}$ (see for instance [5] or [6]):

$$\Sigma_x(k) \mid +\rangle_k = \mid -\rangle_k, \qquad \Sigma_x(k) \mid -\rangle_k = \mid +\rangle_k, \tag{30}$$

$$\Sigma_y(k) \mid +\rangle_k = i \mid -\rangle_k, \quad \Sigma_y(k) \mid -\rangle_k = -i \mid +\rangle_k. \tag{31}$$

Then, $\Psi(x_1,x_2,x_3)$ is:

- (α) An eigenvector for each one of $\Sigma_x(1)\Sigma_y(2)\Sigma_y(3)$, $\Sigma_y(1)\Sigma_x(2)\Sigma_y(3)$, and $\Sigma_y(1)\Sigma_y(2)\Sigma_x(3)$ with eigenvalue 1,
- (β) An eigenvector for $\Sigma_x(1)\Sigma_x(2)\Sigma_x(3)$ with eigenvalue -1.

From (α) and (β) one can easily deduce two facts formalized in Lemma GHZ 1 and Lemma GHZ 2:

Lemma GHZ 1. Assuming locality, the quantities that each can be measured by the $\Sigma_y(k)$'s cannot all have definite values at the time when the measurement of any of them is performed.

Proof of Lemma GHZ 1. For otherwise, one could predict the values $s_x(k)$ that would be measured by all of the $\Sigma_x(k)$'s. But then, denoting the supposedly known values $s_y(k)$ of the measurements $\Sigma_y(k)$, by (α) we would get that:

$$s_x(1)s_y(2)s_y(3) = s_y(1)s_x(2)s_y(3) = s_y(1)s_y(2)s_x(3) = 1, \tag{32}$$

which using:

$$s_y(1)^2 = s_y(2)^2 = s_y(3)^2 = 1, \tag{33}$$

yields:

$$s_x(1)s_x(2)s_x(3) = 1. \tag{34}$$

Since (β) reads:

$$s_x(1)s_x(2)s_x(3) = -1, \tag{35}$$

we can now compare equations (34) and (35). This provides the contradiction that we seek to conclude the proof. More precisely, this proves that not all the $\Sigma_y(k)$'s can have definite values, even if they are unknown, at the time when measurement is performed: the rest is by symmetry over the indices, assuming that all the future decisions are equally possible and locality. **Q.E.D.**

Lemma GHZ 2. The quantities that each can be measured by the $\Sigma_x(k)$'s cannot all have definite values at the time when the measurement of any of them is performed.

Proof of Lemma GHZ 2. For otherwise, denoting the preexisting values of the measurements $\Sigma_x(k)$ respectively by $s_x(k)$, assume that some first measurement $\Sigma_y(j)$ is performed. Without loss of generality, we can assume that $j = 1$, with a result $s_y(1)$ for the measurement. Then, by (α) we would be able to predict with certainty the values $s_y(2)$ and $s_y(3)$ respectively for the measurements $\Sigma_y(2)$ and $\Sigma_y(3)$, from which the same contradiction as obtained above for the previous lemma follows readily. This proves that not all the $\Sigma_x(k)$'s can have definite values, even if they are unknown, at the time when measurement is performed: the rest is by symmetry over the indices, assuming again that all the future decisions are equally possible and locality. **Q.E.D.**

These two lemmas are enough to show that weak realism and locality cannot both hold true in general in the GHZ context. We notice that a small modification of the arguments yields the following stronger result (one just has to consider all possible values of potentially existing quantities which make the proof longer but not otherwise different).

GHZ theorem. In the context of 3-particles GHZ, the quantities that each can be measured by the $\Sigma_y(k)$'s and the $\Sigma_x(k)$'s cannot all have a defined value at the time when the measurement of any of them is performed, or locality fails to be true. Thus one at least of (weak) realism and locality must be wrong.

Like in the case of the EPRB entanglement (see e.g., [11]), experiments have been done on the GHZ entanglement (see e.g., [30]). Some statistical analysis done on the GHZ experiments, which use the less than perfect performance of the captors, such as [31,32] and the critical papers responding to these attacks will not be considered here, and neither will other entanglements such as in [33].

4.3 GHZ with no locality assumption

The particularly great value of GHZ in the overall objective of the present paper is that one can deal with the original three particle version of GHZ and simply replace locality by the EACP and still obtain the same contradiction. In contrast, while dealing with Bell's inequalities we had to find a very special example to accommodate for the substitution of hypotheses.

GHZ condemns realism or the EACP theorem. Under the usual settings of GHZ, all the equations derived assuming locality remain true by assuming the EACP instead. As a consequence, at least one of the EACP and (weak) realism is proven wrong.

Proof of the GHZ condemns realism or the EACP theorem. With A, B, and C standing for the three locations where measurements are made on each of the three particles, we choose three Lorentz observers:

(1) An A-(B, C) observer for whom what happens at A precedes what is going on at both B and C for any triplet and for whom the measurement at B and C on elements of the triplets are seen as simultaneous.
(2) A B-(C, A) observer for whom what happens at B precedes what is going on at both C for any triplet and A and for whom the measurement at C and A on elements of the triplets are seen as simultaneous.
(3) A C-(A, B) observer for whom what happens at C precedes what is going on at both A and B for any triplet and for whom the measurement at A and B on elements of the triplets are seen as simultaneous.

These Lorentz observers take each the two spin projections measurements (or unknown but well defined values deduced from weak realism) at the station that they see first and then compare notes and obtain the same contradiction as in the derivation in Section 4.2 right above where one assumes locality. **Q.E.D.**

With the view that a GHZ theorem is a Bell's theorem without inequalities, this new analysis of GHZ entanglement where we do not assume locality only reinforces the conclusion of the corollary in Section 3.4. For this author it is one more "proof" that weak realism cannot hold true at small enough scale (see Rem. 3 in Sect. 3.3).

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