Transfer-Matrix Method

$k_x^2 + k_z^2 = \omega^2 \mu_0 \epsilon + i \mu_0 \sigma \omega = \left(\frac{\omega}{c}\right)^2 n^2,$ where n is the refractive index, giving

$k'_x = \sqrt{\epsilon_r + i\frac{\sigma}{\epsilon_0 \omega} - k'^2_z}$ where $k = \frac{\omega}{c} k'$, $k'_z = n_0 \sin(\theta),$ (n_0 is n for the first layer)

$k'_x = \sqrt{n^2 - k'^2_z}$

TE-polarisation (E parallel to interface, i.e. $E_x = E_z = 0$; $H_y = 0$; only consider tangential H_z term)

$E_y = (A e^{-ikx} + B e^{ik_x x}) e^{-i\omega t} e^{ik_z z}$

$\vec{H} = \frac{-i}{\mu_0 \omega} \vec{\nabla} \times \vec{E}$ i.e. $\begin{pmatrix} E_y(x) \\ H_z(x) \end{pmatrix} = \begin{pmatrix} e^{-ik_x x} & e^{ik_x x} \\ \frac{-k_x}{\omega \mu_0} e^{-ik_x x} & \frac{k_x}{\omega \mu_0} e^{ik_x x} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$

$H_z = \frac{k_x}{\mu_0 \omega} (-A e^{-ik_x x} + B e^{ik_x x})$

Multiply H terms by $c\mu_0$ to get $\begin{pmatrix} E_y(x) \\ c B_z(x) \end{pmatrix} = \begin{pmatrix} e^{-ik_x x} & e^{ik_x x} \\ -k'_x e^{-ik_x x} & k'_x e^{ik_x x} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = m\vec{b},$ which gives

$$\vec{b}_{j+1} = \begin{pmatrix} e^{-ik_{j+1} x} & e^{ik_{j+1} x} \\ -k'_{j+1} e^{-ik_{j+1} x} & k'_{j+1} e^{ik_{j+1} x} \end{pmatrix}^{-1} \begin{pmatrix} e^{-ik_j x} & e^{ik_j x} \\ -k'_j e^{-ik_j x} & k'_j e^{ik_j x} \end{pmatrix} \vec{b}_j = m_{j+1}^{-1} m_j \vec{b}_j$$

$\vec{b}_{j+1} = M_j \vec{b}_j$ i.e. $\vec{b}_R = M_j M_{j-1} \ldots M_1 \vec{b}_L$ for j interfaces

$\begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix}$ giving $r = \frac{-M_{12}}{M_{11}}$, $t = M_{22} - \frac{M_{21} M_{12}}{M_{11}}$ and $R = |r|^2$, $T = |t|^2$

TM-polarisation (H parallel to interface, i.e. $H_x = H_z = 0$; $E_y = 0$; only consider tangential E_z term)

$H_y = (A e^{-ik_x x} + B e^{ik_x x}) e^{i(k_z z - \omega t)}$

$\vec{\nabla} \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} = (\sigma - i\omega\epsilon) \vec{E}$ i.e. $\begin{pmatrix} H_y(x) \\ E_z(x) \end{pmatrix} = \begin{pmatrix} e^{-ik_x x} & e^{ik_x x} \\ \frac{-ik_x e^{-ik_x x}}{(\sigma - i\omega\epsilon)} & \frac{ik_x e^{ik_x x}}{(\sigma - i\omega\epsilon)} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$

$E_z = \frac{i e^{ik_z z}}{(\sigma - i\omega\epsilon)} (-k_x A e^{-ik_x x} + k_x B e^{ik_x x})$

from the top of the page $\sigma - i\omega\epsilon = -i\epsilon_0 \omega n^2$ and $k = \frac{\omega}{c} k'$, and so:

$$\begin{pmatrix} H_y(x) \\ c\epsilon_0 E_z(x) \end{pmatrix} = \begin{pmatrix} e^{-ik_x x} & e^{ik_x x} \\ \frac{k'_x e^{-ik_x x}}{n_x^2} & \frac{-k'_x e^{ik_x x}}{n_x^2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

This means that $\vec{b}_{j+1} = \begin{pmatrix} e^{-ik_{j+1} x} & e^{ik_{j+1} x} \\ \frac{k'_{j+1} e^{-ik_{j+1} x}}{n_{j+1}^2} & \frac{-k'_{j+1} e^{ik_{j+1} x}}{n_{j+1}^2} \end{pmatrix}^{-1} \begin{pmatrix} e^{-ik_j x} & e^{ik_j x} \\ \frac{k'_j e^{-ik_j x}}{n_j^2} & \frac{-k'_j e^{ik_j x}}{n_j^2} \end{pmatrix} \vec{b}_j = M_j \vec{b}_j$

and so on identically as in the TE- case.

P.T.O.

Manual Calculations

Case 1 – R,T for **...n_L=1|n_R=1.5...** for **incident** rays, TE *and* TM at 1550nm

TE:

$$M=\begin{pmatrix}1 & 1\\ -1.5 & 1.5\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}\frac{5}{6} & \frac{1}{6}\\ \frac{1}{6} & \frac{5}{6}\end{pmatrix}$$ (k' = n for incident rays, since $k_z = 0$)

$$R=\left|\frac{1/6}{5/6}\right|^2=0.04$$

$$T=\left|\frac{5}{6}-\frac{(1/6)(1/6)}{5/6}\right|^2=0.64$$

which both match my program's output

TM:

$$M=\begin{pmatrix}1 & 1\\ \frac{1.5}{1.5^2} & \frac{-1.5}{1.5^2}\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ \frac{1}{1^2} & \frac{-1}{1^2}\end{pmatrix}=\begin{pmatrix}1 & 1\\ \frac{1}{1.5} & \frac{-1}{1.5}\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}\frac{5}{4} & \frac{-1}{4}\\ \frac{-1}{4} & \frac{5}{4}\end{pmatrix},$$

$$R=\left|\frac{-(-1/4)}{5/4}\right|^2=0.04$$

$$T=\left|\frac{5}{4}-\frac{(-1/4)(-1/4)}{5/4}\right|^2=1.44$$

which both match my program's output
(Note that $R_{TM} = R_{TE}$, $T_{TM} \neq T_{TE}$, and that $T_{TM} > 1$)

Case 2 – R,T for **...n_L|n_R...** for **incident** rays, TE and TM at 1550nm, varying n_R/n_L
Note: using different ratios for TE- and TM- ; ratios which demonstrate T>1 for each polarisation

TE, n_L=2, n_R=1

$$M=\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ -2 & 2\end{pmatrix}=\begin{pmatrix}\frac{3}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{3}{2}\end{pmatrix},$$

$$R=\left|\frac{-(-1/2)}{(3/2)}\right|^2=\frac{1}{9}$$

$$T=\left|\frac{3}{2}-\frac{(-1/2)(-1/2)}{3/2}\right|^2=1\frac{7}{9}$$

both of which match my program's output

TM, n_L=1, n_R = 2

$$M=\begin{pmatrix}1 & 1\\ \frac{2}{2^2} & \frac{-2}{2^2}\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ \frac{1}{1^2} & \frac{-1}{1^2}\end{pmatrix}=\begin{pmatrix}1 & 1\\ \frac{1}{2} & \frac{-1}{2}\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}\frac{3}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{3}{2}\end{pmatrix},$$

$$R=\left|\frac{-(-1/2)}{(3/2)}\right|^2=\frac{1}{9}$$

$$T=\left|\frac{3}{2}-\frac{(-1/2)(-1/2)}{3/2}\right|^2=1\frac{7}{9}$$

which both match my program's output
(Note that T > 1 for TE- and TM- polarisations)