

IIII Fermat's Theorem is named after Pierre Fermat (1601–1665), a French lawyer who took up mathematics as a hobby. Despite his amateur status, Fermat was one of the two inventors of analytic geometry (Descartes was the other). His methods for finding tangents to curves and maximum and minimum values (before the invention of limits and derivatives) made him a forerunner of Newton in the creation of differential calculus.

4 Fermat's Theorem If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Proof Suppose, for the sake of definiteness, that f has a local maximum at c . Then, according to Definition 2, $f(c) \geq f(x)$ if x is sufficiently close to c . This implies that if h is sufficiently close to 0, with h being positive or negative, then

$$f(c) \geq f(c + h)$$

and therefore

$$\text{5} \quad f(c + h) - f(c) \leq 0$$

We can divide both sides of an inequality by a positive number. Thus, if $h > 0$ and h is sufficiently small, we have

$$\frac{f(c + h) - f(c)}{h} \leq 0$$

Taking the right-hand limit of both sides of this inequality (using Theorem 2.3.2), we get

$$\lim_{h \rightarrow 0^+} \frac{f(c + h) - f(c)}{h} \leq \lim_{h \rightarrow 0^+} 0 = 0$$

But since $f'(c)$ exists, we have

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \rightarrow 0^+} \frac{f(c + h) - f(c)}{h}$$

and so we have shown that $f'(c) \leq 0$.

If $h < 0$, then the direction of the inequality (5) is reversed when we divide by h :

$$\frac{f(c + h) - f(c)}{h} \geq 0 \quad h < 0$$

upward $h \neq$ downward h
so this statement is wrong

So, taking the left-hand limit, we have

$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} = \lim_{h \rightarrow 0^-} \frac{f(c + h) - f(c)}{h} \geq 0$$

We have shown that $f'(c) \geq 0$ and also that $f'(c) \leq 0$. Since both of these inequalities must be true, the only possibility is that $f'(c) = 0$.

We have proved Fermat's Theorem for the case of a local maximum. The case of a local minimum can be proved in a similar manner, or we could use Exercise 76 to deduce it from the case we have just proved (see Exercise 77).

The following examples caution us against reading too much into Fermat's Theorem. We can't expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x .

EXAMPLE 5 If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0, as you can see from its graph in Figure 9. (Or observe that $x^3 > 0$ for $x > 0$ but $x^3 < 0$ for $x < 0$.) The fact that $f'(0) = 0$ simply means that the curve $y = x^3$ has a

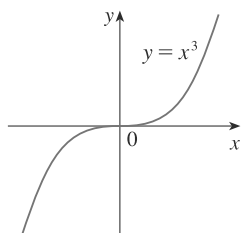


FIGURE 9

If $f(x) = x^3$, then $f'(0) = 0$ but f has no maximum or minimum.