CALCULUS III
WORKSHEET

Section 4-2 : Iterated Integrals

1. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to x first and **(b)** by integrating with respect to y first.

$$\iint\limits_R 16xy - 9x^2 + 1\, dA \qquad R = [2,3] \times [-1,1]$$

2. Compute the following double integral over the indicated rectangle **(a)** by integrating with respect to x first and **(b)** by integrating with respect to y first.

$$\iint\limits_R \cos(x)\sin(y)\, dA \qquad R = \left[\tfrac{\pi}{6}, \tfrac{\pi}{4}\right] \times \left[\tfrac{\pi}{4}, \tfrac{\pi}{3}\right]$$

For problems 3 – 16 compute the given double integral over the indicated rectangle.

3. $\displaystyle\iint\limits_R 8x^3 - 4\, dA \qquad R = [-3,-1] \times [0,4]$

4. $\displaystyle\iint\limits_R 15y^4 + \frac{2}{x^3}\, dA \qquad R = [1,2] \times [1,4]$

5. $\displaystyle\iint\limits_R 4y\sec^2(x) + \frac{2x}{y}\, dA \qquad R = \left[0, \tfrac{\pi}{4}\right] \times [1,5]$

6. $\displaystyle\iint\limits_R y^5 - x^2\mathbf{e}^y\, dA \qquad R = [-1,2] \times [-3,3]$

7. $\displaystyle\iint\limits_R \frac{x^3}{1+x^4} - \frac{1}{\mathbf{e}^{3y}}\, dA \qquad R = [-1,0] \times [0,4]$

8. $\displaystyle\iint\limits_R x\mathbf{e}^{x^2} - 12x^3\sin(\pi y)\, dA \qquad R = [-2,0] \times \left[\tfrac{1}{2}, 1\right]$

9. $\displaystyle\iint\limits_R x\cos\left(4y + 3x^2\right) dA \qquad R = \left[0, \sqrt{\pi}\right] \times \left[\tfrac{\pi}{2}, \pi\right]$

10. $\displaystyle\iint\limits_R \frac{\ln(4xy)}{xy}\, dA \qquad R = [1,2] \times [3,4]$

Section 4-4 : Double Integrals in Polar Coordinates

1. Evaluate $\iint\limits_D 3xy^2 - 2\,dA$ where D is the unit circle centered at the origin.

2. Evaluate $\iint\limits_D 4x - 2y\,dA$ where D is the top half of region between $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

3. Evaluate $\iint\limits_D 6xy + 4x^2\,dA$ where D is the portion of $x^2 + y^2 = 9$ in the 2nd quadrant.

4. Evaluate $\iint\limits_D \sin\left(3x^2 + 3y^2\right)dA$ where D is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 7$.

5. Evaluate $\iint\limits_D \mathbf{e}^{1-x^2-y^2}\,dA$ where D is the region in the 4th quadrant between $x^2 + y^2 = 16$ and $x^2 + y^2 = 36$.

6. Use a double integral to determine the area of the region that is inside $r = 6 - 4\cos\theta$.

7. Use a double integral to determine the area of the region that is inside $r = 4$ and outside $r = 8 + 6\sin\theta$.

8. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_{-2}^{0}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2\,dx\,dy$$

9. Evaluate the following integral by first converting to an integral in polar coordinates.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$$

10. Use a double integral to determine the volume of the solid that is below $z = 9 - 4x^2 - 4y^2$ and above the xy-plane.

11. Use a double integral to determine the volume of the solid that is bounded by $z = 12 - 3x^2 - 3y^2$ and $z = x^2 + y^2 - 8$.

12. Use a double integral to determine the volume of the solid that is inside both the cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 16$.

13. Use a double integral to derive the area of a circle of radius a.

Section 4-5 : Triple Integrals

1. Evaluate $\int_{1}^{2}\int_{0}^{2}\int_{-1}^{1} 2+z^2-xy\,dz\,dx\,dy$

2. Evaluate $\int_{2}^{0}\int_{x^2}^{1}\int_{0}^{xz} y^2-6z\,dy\,dz\,dx$

3. Evaluate $\int_{-1}^{2}\int_{0}^{1}\int_{0}^{2z} 3x-\sqrt{1+z^2}\,dx\,dz\,dy$

4. Evaluate $\iiint\limits_E 12y\,dV$ where E is the region below $6x+4y+3z=12$ in the first octant.

5. Evaluate $\iiint\limits_E 5x^2\,dV$ where E is the region below $x+2y+4z=8$ in the first octant.

6. Evaluate $\iiint\limits_E 10z^2-x\,dV$ where E is the region below $z=8-y$ and above the region in the *xy*-plane bounded by $y=2x$, $x=3$ and $y=0$.

7. Evaluate $\iiint\limits_E 4y^2\,dV$ where E is the region below $z=-3x^2-3y^2$ and above $z=-12$.

8. Evaluate $\iiint\limits_E 2y-9z\,dV$ where E is the region behind $6x+3y+3z=15$ front of the triangle in the *xz*-plane with vertices, in (x,z) form : $(0,0)$, $(0,4)$ and $(2,4)$.

9. Evaluate $\iiint\limits_E 18x\,dV$ where E is the region behind the surface $y=4-x^2$ that is in front of the region in the *xz*-plane bounded by $z=-3x$, $z=2x$ and $z=2$.

10. Evaluate $\iiint\limits_E 20x^3\,dV$ where E is the region bounded by $x=2-y^2-z^2$ and $x=5y^2+5z^2-6$.

11. Evaluate $\iiint\limits_E 6z^2\,dV$ where E is the region behind $x+6y+2z=8$ that is in front of the region in the *yz*-plane bounded by $z=2y$ and $z=\sqrt{4y}$.

12. Evaluate $\iiint\limits_E 8y\,dV$ where E is the region between $x+y+z=6$ and $x+y+z=10$ above the triangle in the *xy*-plane with vertices, in (x,y) form : $(0,0)$, $(1,2)$ and $(1,4)$.

Section 4-6 : Triple Integrals in Cylindrical Coordinates

1. Evaluate $\iiint\limits_E 8z\,dV$ where E is the region bounded by $z = 2x^2 + 2y^2 - 4$ and $z = 5 - x^2 - y^2$ in the 1st octant.

2. Evaluate $\iiint\limits_E 6xy\,dV$ where E is the region above $z = 2x - 10$, below $z = 2$ and inside the cylinder $x^2 + z^2 = 4$.

3. Evaluate $\iiint\limits_E 9yz^3\,dV$ where E is the region between $x = -\sqrt{9y^2 + 9z^2}$ and $x = \sqrt{y^2 + z^2}$ inside the cylinder $y^2 + z^2 = 1$.

4. Evaluate $\iiint\limits_E x + 2\,dV$ where E is the region bounded by $x = 18 - 4y^2 - 4z^2$ and $x = 2$ with $z \geq 0$.

5. Evaluate $\iiint\limits_E x + 2\,dV$ where E is the region between the two planes $2x + y + z = 6$ and $6x + 3y + 3z = 12$ inside the cylinder $x^2 + z^2 = 16$.

6. Evaluate $\iiint\limits_E x^2\,dV$ where E is the region bounded by $y = x^2 + z^2 - 4$ and $y = 8 - 5x^2 - 5z^2$ with $x \leq 0$.

7. Use a triple integral to determine the volume of the region bounded by $z = \sqrt{x^2 + y^2}$, and $z = x^2 + y^2$ in the 1st octant.

8. Use a triple integral to determine the volume of the region bounded by $y = \sqrt{9x^2 + 9z^2}$, and $y = -3x^2 - 3z^2$ in the 1st octant.

9. Use a triple integral to determine the volume of the region behind $x = z + 3$, in front of $x = -z - 6$ and inside the cylinder $y^2 + z^2 = 4$.

10. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^4 \int_0^{\sqrt{16-y^2}} \int_0^{6+x} 6yx^2\,dz\,dx\,dy$$

11. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{2x^2+2y^2}}^{6+x^2+y^2} 15z\,dz\,dy\,dx$$

Section 4-7 : Triple Integrals in Spherical Coordinates

1. Evaluate $\iiint\limits_E 4y^2\,dV$ where E is the sphere $x^2+y^2+z^2=9$.

2. Evaluate $\iiint\limits_E 3x-2y\,dV$ where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ with $z\le 0$.

3. Evaluate $\iiint\limits_E 2yz\,dV$ where E is the region below $x^2+y^2+z^2=16$ and inside $z=\sqrt{3x^2+3y^2}$ that is in the 1st octant.

4. Evaluate $\iiint\limits_E z^2\,dV$ where E is the region between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=25$ and inside $z=-\sqrt{\frac{1}{3}x^2+\frac{1}{3}y^2}$.

5. Evaluate $\iiint\limits_E 5y^2\,dV$ where E is the portion of $x^2+y^2+z^2=4$ with $x\le 0$.

6. Evaluate $\iiint\limits_E 2+16x\,dV$ where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ with $y\ge 0$ and $z\le 0$.

7. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_0^2\int_{-\sqrt{4-x^2}}^{0}\int_{\sqrt{5x^2+5y^2}}^{\sqrt{6-x^2-y^2}} 7x\;dz\,dy\,dx$$

8. Evaluate the following integral by first converting to an integral in cylindrical coordinates.

$$\int_{-\sqrt{5}}^{\sqrt{5}}\int_0^{\sqrt{5-y^2}}\int_{-\sqrt{18-x^2-y^2}}^{-\sqrt{x^2+y^2}} 3xz^2\;dz\,dx\,dy$$

9. Use a triple integral in spherical coordinates to derive the volume of a sphere with radius a.

Section 4-8 : Change of Variables

For problems 1 – 4 compute the Jacobian of each transformation.

1. $x = 4u^2 v \qquad y = 6v - 7u$

2. $x = \sqrt{u} \qquad y = 10u + v$

3. $x = v^3 u \qquad y = \dfrac{u^2}{v}$

4. $x = \mathbf{e}^u \cos v \qquad y = \mathbf{e}^u \sin v$

5. If R is the region inside $\dfrac{x^2}{25} + 49y^2 = 1$ determine the region we would get applying the transformation $x = 5u$, $y = \dfrac{1}{7}v$ to R.

6. If R is the triangle with vertices $(2,0)$, $(6,4)$ and $(1,4)$ determine the region we would get applying the transformation $x = \dfrac{1}{5}(u - v)$, $y = \dfrac{1}{5}(u + 4v)$ to R.

7. If R is the parallelogram with vertices $(0,0)$, $(4,2)$, $(0,4)$ and $(-4,2)$ determine the region we would get applying the transformation $x = u - v$, $y = \dfrac{1}{2}(u + v)$ to R.

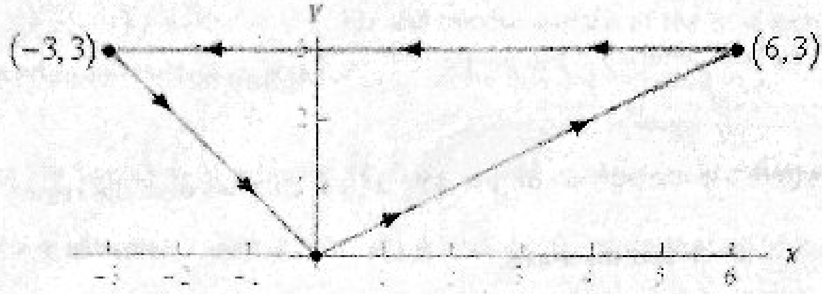
8. If R is the square defined by $0 \le x \le 3$ and $0 \le y \le 3$ determine the region we would get applying the transformation $x = 3u$, $y = v\left(2 + u^2\right)$ to R.

9. If R is the parallelogram with vertices $(1,1)$, $(5,3)$, $(8,8)$ and $(4,6)$ determine the region we would get applying the transformation $x = \dfrac{6}{7}(u - v)$, $y = \dfrac{1}{7}(10u - 3v)$ to R.

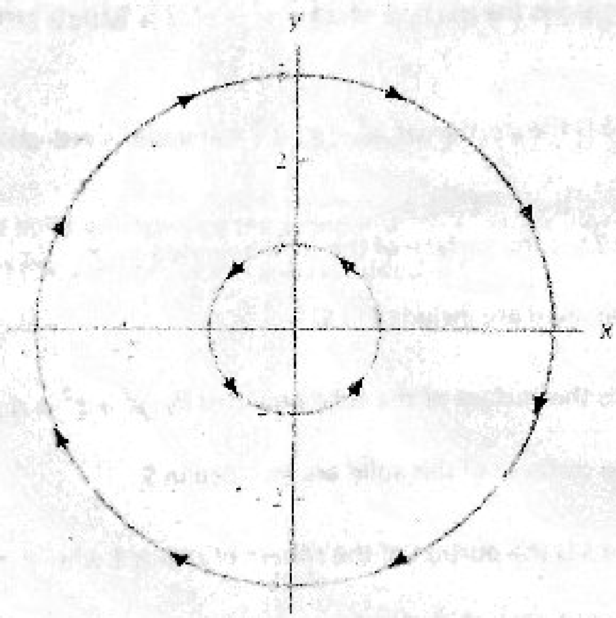
10. If R is the region bounded by $xy = 4$, $xy = 10$, $y = x$ and $y = 6x$ determine the region we would get applying the transformation $x = 2\sqrt{\dfrac{u}{v}}$, $y = 4\sqrt{uv}$ to R.

Section 5-7 : Green's Theorem

1. Use Green's Theorem to evaluate $\int\limits_C \left(yx^2 - y\right)dx + \left(x^3 + 4\right)dy$ where C is shown below.



2. Use Green's Theorem to evaluate $\int\limits_C \left(7x + y^2\right)dy - \left(x^2 - 2y\right)dx$ where C is are the two circles as shown below.



3. Use Green's Theorem to evaluate $\int\limits_C \left(y^3 - 6y\right)dx + \left(y^3 + 10y^2\right)dy$ where C is shown below.

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Section 6-3 : Surface Integrals

1. Evaluate $\iint\limits_S 2x-3y+z\,dS$ where S is the portion of $x+y+z=2$ that is in the 1st octant.

2. Evaluate $\iint\limits_S x+y^2+z^2\,dS$ where S is the portion of $x=4-y^2-z^2$ that lies in front of $x=-2$.

3. Evaluate $\iint\limits_S 6\,dS$ where S is the portion of $y=4z+x^3+6$ that lies over the region in the xz-plane with bounded by $z=x^3$, $x=1$ and the x-axis.

4. Evaluate $\iint\limits_S xyz\,dS$ where S is the portion of $x^2+y^2=36$ between $z=-3$ and $z=1$.

5. Evaluate $\iint\limits_S z^2+x\,dS$ where S is the portion of $x^2+y^2+z^2=4$ with $z\geq 0$.

6. Evaluate $\iint\limits_S 4y\,dS$ where S is the portion of $x^2+z^2=9$ between $y=2$ and $y=10-x$.

7. Evaluate $\iint\limits_S z+3\,dS$ where S is the surface of the solid bounded by $z=2x^2+2y^2-3$ and $z=1$. Note that both surfaces of this solid are included in S.

8. Evaluate $\iint\limits_S z\,dS$ where S is the surface of the solid bounded by $y^2+z^2=4$, $x=y-3$, and $x=6-z$. Note that all three surfaces of this solid are included in S.

9. Evaluate $\iint\limits_S 4+z\,dS$ where S is the portion of the sphere of radius 1 with $z\geq 0$ and $x\leq 0$. Note that all three surfaces of this solid are included in S.

Section 6-6 : Divergence Theorem

1. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=\left(3x-zx^2\right)\vec{i}+\left(x^3-1\right)\vec{j}+\left(4y^2+x^2z^2\right)\vec{k}$ and S is the surface of the box with $0\le x\le 1$, $-3\le y\le 0$ and $-2\le z\le 1$. Note that all six sides of the box are included in S.

2. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=4x\,\vec{i}+\left(1-6y\right)\vec{j}+z^3\,\vec{k}$ and S is the surface of the sphere of radius 2 with $z\ge 0$, $y\le 0$ and $x\ge 0$. Note that all four surfaces of this solid are included in S.

3. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=-xy\,\vec{i}+\left(z-1\right)\vec{j}+z^3\,\vec{k}$ and S is the surface of the solid bounded by $y=4x^2+4z^2-1$ and the plane $y=7$. Note that both of the surfaces of this solid included in S.

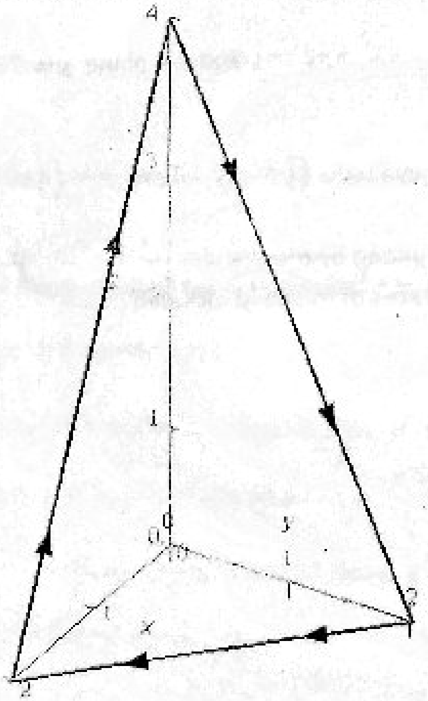
4. Use the Divergence Theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where $\vec{F}=\left(4x-z^2\right)\vec{i}+\left(x+3z\right)\vec{j}+\left(6-z\right)\vec{k}$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=36$ and the planes $z=-2$ and $z=3$. Note that both of the surfaces of this solid included in S.

Section 6-5 : Stokes' Theorem

1. Use Stokes' Theorem to evaluate $\iint\limits_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = x^3\,\vec{i} + \left(4y - z^3y^3\right)\vec{j} + 2x\,\vec{k}$ and S is the portion of $z = x^2 + y^2 - 3$ below $z = 1$ with orientation in the negative z-axis direction.

2. Use Stokes' Theorem to evaluate $\iint\limits_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = 2y\,\vec{i} + 3x\,\vec{j} + \left(z - x\right)\vec{k}$ and S is the portion of $y = 11 - 3x^2 - 3z^2$ in front of $y = 5$ with orientation in the positive y-axis direction.

3. Use Stokes' Theorem to evaluate $\int\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left(zx^3 - 2z\right)\vec{i} + xz\,\vec{j} + yx\,\vec{k}$ and C is the triangle with vertices $(0,0,4)$, $(0,2,0)$ and $(2,0,0)$. C has a clockwise rotation if you are above the triangle and looking down towards the xy-plane. See the figure below for a sketch of the curve.



4. Use Stokes' Theorem to evaluate $\int\limits_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\,\vec{i} - 4z\,\vec{j} + xy\,\vec{k}$ and C is is the circle of radius 1 at $x = -3$ and perpendicular to the x-axis. C has a counter clockwise rotation if you are looking down the x-axis from the positive x-axis to the negative x-axis. See the figure below for a sketch of the curve.