4. The canonical integral. The equations (63.5) have a double origin. The first set of equations holds on account of Legendre's transformation and can be considered as an implicit definition of the momenta p_i. The second set of equations is a consequence of the variational principle. Yet the conspicuous symmetry of the complete set of equations suggests that they must be deducible from one single principle. This is actually the case.

Because of the duality of Legendre's transformation, we can start with the Hamiltonian function H and construct the Lagrangian function L. We obtain L by means of the relation

$$L = \sum_{i=1}^{n} p_i \dot{q}_i - H. \tag{64.1}$$

We then have to eliminate the p_i by expressing them as functions of the q_i and $\dot{q}_i$. Closer inspection shows, however, that this elimination *need not be performed.*

Let us investigate how the variation of the p_i influences the variation of the action integral. The variation of (64.1) with respect to the p_i gives

$$\delta L = \sum_{i=1}^{n} \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i, \tag{64.2}$$

but the coefficient of δp_i is zero on account of Legendre's transformation. This shows that *an arbitrary variation of the p_i has no influence on the variation of L.* But then it has no influence either on the integral of L with respect to the time. This gives the following important result. Originally we required the vanishing of the first variation of the action integral L, with the restriction that the p_i are not free variables but certain given functions of the q_i and $\dot{q}_i$. The variation of p_i is thus determined by the variation of q_i. However, since the variation of p_i has no influence on the variation of the action integral, we can enlarge the validity of the original variational principle and state that the action integral assumes a stationary value *even if the p_i are varied arbitrarily,* which means *even if the p_i are considered as a second set of independent variables.*

Hence it is not necessary to change anything in the Lagrangian function (64.1). We can form the action integral

$$A = \int_{t_1}^{t_2} [\Sigma\, p_i\dot{q}_i - H(q_1, \ldots, q_n; p_1, \ldots, p_n; t)]\, dt, \tag{64.3}$$

and require that it assume a stationary value for arbitrary variations of the q_i *and* the p_i. This new variation problem has $2n$ variables and the Euler-Lagrange differential equations can be formed with respect to all the q_i and the p_i. If we do that, we get the following $2n$ differential equations:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} &\equiv \frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} = 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{p}_i} - \frac{\partial L}{\partial p_i} &\equiv 0 - \dot{q}_i + \frac{\partial H}{\partial p_i} = 0. \end{aligned} \tag{64.4}$$

These are exactly the canonical equations (63.5), *now conceived as a unified system of* $2n$ *differential equations, derived from the action integral* (64.3). We no longer need the original Lagrangian function and the Legendre transformation by which H was obtained. We have a new variational principle which is equivalent to the original principle, yet superior on account of the simpler structure of the resulting differential equations. These equations are no longer of the second order, but of the first. The derivatives are all separated and not intermingled with algebraic operations.

This remarkable simplification is achieved on account of the simple form of the new action integral (64.3). We call this form the "canonical integral." The integrand has again the form, "kinetic energy minus potential energy," since the second term of the integrand is a mere function of the position coordinates—which are now the q_i *and* the p_i—while the first term depends on the velocities. It is the ***kinetic part*** of the canonical integral which accounts for its remarkable properties. The "kinetic energy" is now a *simple linear function of the velocities* $\dot{q}_i$,[1] namely:

$$p_1\dot{q}_1 + p_2\dot{q}_2 + \ldots + p_n\dot{q}_n. \tag{64.5}$$

[1]In classical mechanics the kinetic energy happens to be a quadratic function of the velocities. This, however, is by no means necessary. The transformation to the canonical form can be performed for arbitrary Lagrangian functions, however complicated.