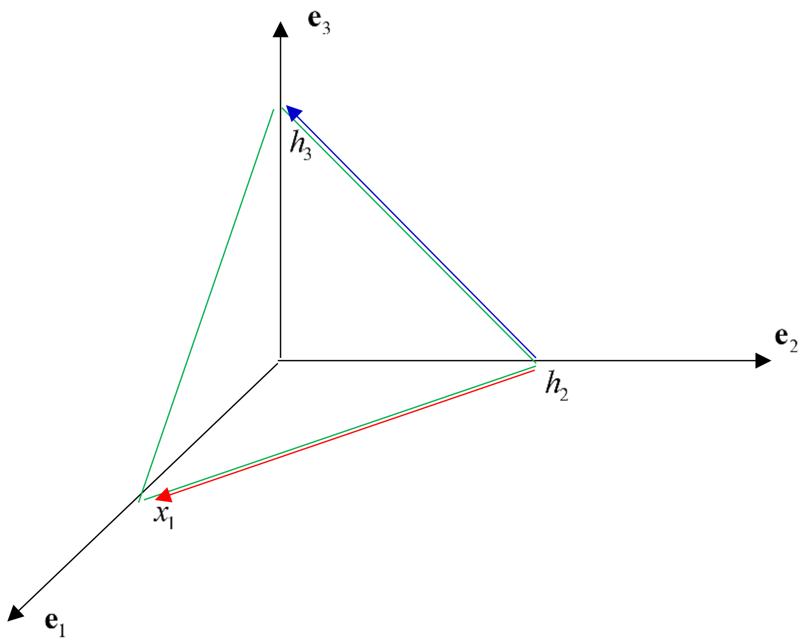
CAUCHY STRESS TETRAHEDRON



Let h be the distance from the origin to the oblique face defined by: h1, h2, h3

Let n be the unit vector normal to that face.

The vector, **n**, can be expressed as

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$$

Thus the components are

$$n_k = \frac{h}{h_k}$$

Now I need two more vectors

RED VECTOR: $\mathbf{r} = h_1\mathbf{e}_1 - h_2\mathbf{e}_2$

BLUE VECTOR: $\mathbf{b} = -h_2\mathbf{e}_2 + h_3\mathbf{e}_3$

The area of the green triangle is one half the magnitude of the cross product of the red and blue.

$$\textbf{Cross} = \left(h_1\mathbf{e}_1 - h_2\mathbf{e}_2\right) \times \left(-h_2\mathbf{e}_2 + h_3\mathbf{e}_3\right)$$

Thus

$$\textbf{Cross} = -h_2h_3\mathbf{e}_1 - h_1h_3\mathbf{e}_2 - h_1h_2\mathbf{e}_3$$

$$A = \|\mathbf{Cross}\| = \frac{1}{2}\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}$$

Next, we would like the volume of the tetrahedron. In all four cases, it will be the face area times the height.

$$V = \frac{1}{3}Ah = \frac{1}{3}A_1h_1 = \frac{1}{3}A_2h_2 = \frac{1}{3}A_3h_3$$

In addition, the areas of the three faces in the Cartesian plane are

$$A_1 = \frac{1}{2}h_2h_3$$

$$A_2 = \frac{1}{2}h_1h_3$$

$$A_3 = \frac{1}{2}h_1h_2$$

Now begin with $V = \frac{1}{3}Ah$

Thus

$$\frac{3V}{A} = h$$

$$h = \frac{3V}{\frac{1}{2}\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

$$h = \frac{A_1h_1}{\frac{1}{2}\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

$$h = \frac{\frac{1}{2}h_1h_2h_3}{\frac{1}{2}\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

$$h = \frac{h_1h_2h_3}{\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

$$h = \frac{h_1h_2h_3}{2A}$$

Next, I need a better expression for the unit normal to A.

Begin with the cross product we found and normalize it.

$$\textbf{Cross} = -h_2h_3\mathbf{e}_1 - h_1h_3\mathbf{e}_2 - h_1h_2\mathbf{e}_3$$

Now normalize the cross to get the vector, **n**

$$\mathbf{n} = \pm\frac{-h_2h_3\mathbf{e}_1 - h_1h_3\mathbf{e}_2 - h_1h_2\mathbf{e}_3}{\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

But choose the positive out

$$\mathbf{n} = \frac{h_2h_3\mathbf{e}_1 + h_1h_3\mathbf{e}_2 + h_1h_2\mathbf{e}_3}{\sqrt{(h_2h_3)^2 + (h_1h_3)^2 + (h_1h_2)^2}}$$

Now revert back to the Area

$$\mathbf{n} = \frac{h_2h_3\mathbf{e}_1 + h_1h_3\mathbf{e}_2 + h_1h_2\mathbf{e}_3}{2A}$$

But we found

$$h = \frac{h_1h_2h_3}{2A}$$

Or

$$2A = \frac{h_1h_2h_3}{h}$$

Thus

$$\mathbf{n} = \frac{h_2h_3\mathbf{e}_1 + h_1h_3\mathbf{e}_2 + h_1h_2\mathbf{e}_3}{\left(\dfrac{h_1h_2h_3}{h}\right)}$$

$$\mathbf{n} = \left(\frac{h}{h_1}\right)\mathbf{e}_1 + \left(\frac{h}{h_2}\right)\mathbf{e}_2 + \left(\frac{h}{h_3}\right)\mathbf{e}_3$$

We can now return to $V = \frac{1}{3}Ah = \frac{1}{3}A_1h_1 = \frac{1}{3}A_2h_2 = \frac{1}{3}A_3h_3$

We can pull out one

$$\frac{1}{3}Ah = \frac{1}{3}A_1h_1$$

$$\frac{A_1}{A} = \frac{h}{h_1}$$

And we find

$$\frac{A_1}{A} = \frac{h}{h_1}$$

And that is the direction cosine to give the area formulas