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A CENSORED-GARCH MODEL OF ASSET RETURNS WITH PRICE LIMITS

Steven X. Wei[1]

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Abstract

As one important form of market circuit breakers, *price limits* have been often imposed in stock and futures markets. This paper considers modeling the return process of such assets, focusing on the treatment of price limits. As a result, a *censored-GARCH model* is formulated and a Bayesian approach to this model is developed. An application is provided to Treasury bill futures over a period of high volatility and frequent limit moves. The impacts of price limits are demonstrated with the real data and confirmed with a simulation example.

Keywords: Price limits, censored-GARCH model, griddy Gibbs sampler-data augmentation.

JEL classification: C13, C24 and G19

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[1]CORE, Université Catholique de Louvain and Department of Finance, The Hong Kong University of Science and Technology, Email: weix@uxmail.ust.hk.

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1 Introduction

The excess volatility of financial markets has drawn much attention over the past decade. An early and serious consideration of this issue was initiated immediately after the stock market crash in October 1987. As a result, the Brady commission (1988), along with several other commissions and exchanges, recommended setting up *circuit breakers* in financial markets to attenuate their excess volatility. As one important form of circuit breakers, *price limits* have been imposed in many financial markets, such as stock markets in Austria, Belgium, France, Italy, Switzerland, Spain, Japan, Korea and Taiwan; commodity futures exchanges of corn, oats, soybeans, wheat, cotton, gold and silver in the United States; and financial futures exchanges of foreign currencies, US Treasury bonds, notes and bills.

What have we learned about price limits from the literature? In a theoretical domain, Kodres and O'Brien (1994) argued that price limits are Pareto efficient if the implementation risk, a form of market incompleteness, is considered. For futures contracts, Brennan (1986) concluded that price limits are a partial substitute for margin requirements, alleviating the overall cost of trading and reducing the risk of contract default. In an empirical domain, there has been a large amount of research associated with price limits. Using a cross-sectional data analysis, Kim and Rhee (1997) provided empirical evidence on the claim that price limits spillover volatility, delay price discovery and interfere trading. On the other hand, a critical issue involved in the other existing empirical studies is how to model the return process of such assets, which has not been properly studied in the literature and is the focus of the current study.

More specifically, this paper considers modeling the return process of such assets, focusing on the treatment of price limits. A *censored-GARCH model* (GARCH: generalized autoregressive conditional heteroskedasticity) is formulated and a Bayesian approach to estimating this model is developed. A salient feature of this model is its ability to fully capture the constraints on consecutive unobserved equilibrium returns implied by price limits, which distinguishes this article from the existing others. This feature brings a challenge to the estimation of this model. Consequently, the proposed Bayesian approach represents one of the major contributions of this article.

Early empirical work relating to price limits was seen in Hodrick and Srivastava (1987) and McCurdy and Morgan (1987). In their studies, price limits were either ignored or deleted. Ignoring price limits means that the observed market price is taken as if it was the equilibrium price in the event of a limit move. Deleting price limits is to drop the limited prices from the studied sample. Such treatments can result in negative consequences. As pointed out by Wei and Chiang (1997), the standard deviation for Japanese yen futures during 1977-1979 is underestimated by 5.7% when price limits are ignored, and by 14.3% when price limits are deleted. The underestimated volatilities resulted in an underpricing of approximately 10% and 21%, respectively, for an at-the-money call option defined on the Japanese futures, when the option has one year to maturity.[1] In addition, deleting limited prices breaks down the dynamic structure of a price series, and is therefore not recommended. With a fairly significant number of price limit moves, this paper demonstrates that price limits result in the thin-tailedness of the returns of such assets and distort the tail behavior of the returns. These findings are consistent with the results obtained by Yang and Brorsen (1995) for pork bellies futures.

[1] For a reference of futures options, see Hull (1997, pp 273-280).

Another strategy adopted by McCurdy and Morgan (1987) is to lower the frequency of sampling – weekly on Wednesdays. In general, this strategy may fail because price limits can happen on any trading day. In addition, this reduces the sample size substantially and thus lowers the statistical precision (see Sutrick (1993)). Conventional data aggregation can be more harmful to making statistical inference since it introduces bias in the aggregated data before any statistical inference is conducted. Recently, Wei and Chiang (1997) took a rather different approach, in which daily price data were converted into irregularly spaced ones: accumulating consecutive unobserved equilibrium returns and treating the multi-day return as a single unit. The success of this conversion is based on the fact that the observed accumulated holding return is identical to the (unobserved) accumulated equilibrium holding return. Two limitations in their approach appear immediate: (1) it relies on the assumption that price limits place no impact on the underlying (equilibrium) asset's price-generating process; and (2) it is hard to extend their approach to the case of conditional heteroskedasticity.

Two studies by Kodres (1988, 1993) took an important step towards a formal treatment of price limits in the context of econometrics. Although she intended to examine whether price limits affect the testing of an unbiasedness hypothesis in foreign exchange markets,[2] the key idea of her modeling is quite useful: *limited prices are treated as censored variables.*[3] This is understood because when a price limit is reached, the equilibrium price is no longer observable and is beyond the reached limit. Kodres (1988) developed a censored regression model with a lagged latent dependent variable, that was recently developed into dynamic Tobit models by Lee (1997) and Wei (1997). Taking account of conditional heteroskedasticity, Kodres (1993) renewed her previous model and formed a model that later led to the development of Tobit-GARCH models (see, Lee (1997) and Galzolari and Fiorentini (1997)).[4] The main conclusions of her two papers are the same: price limits do not significantly affect the testing of the unbiasedness hypothesis. From a modeling point of view, however, Kodres (1988, 1993) failed to provide a proper structure of censoring for returns. The censored-GARCH model formed in this paper results from the realization of this problem.

Morgan and Trevor (1997) criticized the estimation technique proposed by Kodres (1988, 1993) in two aspects: (1) selective use of forward price approximation of futures price from covered interest rate arbitrage and (2) numerical complexity. Using forward price approximation likely distorts the variance estimates of the parameters since the uncertainty associated with the unobserved equilibrium futures prices is about to be ignored in this treatment. In addition, the approximation treatment is not a sensible method since for some other assets, the forward approximation may not exist at all. The numerical computation in Kodres' estimation method can immediately become practically impossible with the dimensional increase of consecutive unobserved equilibrium prices. Morgan and Trevor (1997) developed a Rational Expectation (RE) method, similar to the approach by Calzolari and Fiorentini (1997), for the estimation of Kodres' (1993) model. Lee (1997) studied a Tobit-ARCH (GARCH) model with the simulated maximum likelihood (SML) method. Although these methods might be fairly easily extended to dealing with a normal version of the censored-GARCH

[2]The unbiasedness hypothesis studied by Kodres (1988, 1993) can be stated as whether today's futures price is an unbiased predictor of tomorrow's spot price.

[3]The original idea of dealing with a censored regression model is attributed to Tobin (1958).

[4]There is no standard notion for the Tobit-ARCH (GARCH) model. Lee (1997) named it the ARCH (GARCH)–Tobit model and Calzolari and Fiorentini (1997) called it the Tobit-ARCH (GARCH) model. In this paper, I follow the notion of Calzolari and Fiorentini.

model, they cannot compete with the proposed method in this paper on the flexibility of choice of thin/fat-tailed conditional error distributions.

A few additional advantages associated with the developed Bayesian method are in order: (1) it is natural and convenient to deal with linear constraints (including truncations as a special case) on both the model parameters and the latent dependent variables; (2) it is flexible to both prior and likelihood specifications; (3) it provides finite sample inference results because it is Bayesian; and (4) it is straightforward to code and implement.

The proposed model and estimation method are applied to Treasury bill futures over a period of high volatility and frequent limit moves. It is found that ignoring price limits results in large distortion on the posterior distributions of the model parameters. This is especially true for the tail-thickness parameter. A simulation example confirms the point and further shows that the censored-GARCH model is indeed a proper description for the asset returns subject to daily price limits. Both the real and simulated data indicate the substantial deviation of the posterior distributions from the normal family.

The remainder of this paper proceeds as follows. Section 2 proposes a censored GARCH model and Section 3 develops a simple and practical posterior estimation method for the model. Section 4 offers an application of the model and the method with T-bill futures data and Section 5 provides a simulation example to confirm the impacts of price limits reported in Section 4. Conclusion is given in Section 6.

2 The model

This section models the return process of assets when price limits are present. Two notions of both price and return are distinguished and linked. As a result, a *censored-GARCH model* is formulated. The prior specification of the model parameters is also discussed.

Usually, daily price limits are set at the previous-day's (closing) price plus and minus a constant, say a.[5] When a price limit is hit, the observed market price being equal to the limit deviates from its equilibrium value. It is crucial to distinguish them in the current study. Let p_t^* and p_t be the market equilibrium and observed prices at time (*i.e.*, day) t, respectively. They are linked in the following non-linear fashion:

$$p_t = \begin{cases} p_{t-1} + a & \text{if} \quad p_t^* \geq p_{t-1} + a \\ p_t^* & \text{if} \quad p_{t-1} - a < p_t^* < p_{t-1} + a \\ p_{t-1} - a & \text{if} \quad p_t^* \leq p_{t-1} - a \end{cases} . \tag{1}$$

In words, the intrinsic value of the price, p_t^*, can be observed only if it stays in a predetermined symmetric band $(p_{t-1} - a,\ p_{t-1} + a)$. The structure linking p_t^* and p_t resembles the one in the literature of limited dependent variables models (See, for example, Tobin (1958) and Maddala (1987)). If a (conditional mean) dynamic structure is imposed in the process of p_t^*, a dynamic Tobit model for p_t can be immediately formed. (For a detailed discussion of dynamic Tobit models, see Lee (1997) from a classical point of view and Wei (1997) from a Bayesian point of view.)

However, most empirical work in finance inclines to model return rather than price itself for three reasons. First, return is a complete and scale-free summary of the investment oppor-

[5]It is stressed that the model proposed in this paper can be easily adapted to the more general case in which a is a (conditional) deterministic process.

tunity. Second, traders are mainly concerned about their investment returns. Third, return has more attractive statistical properties than price, such as symmetry and stationarity.

In a setting without price limits, the conversion of prices into returns is straightforward. However, caution must be exercised when price limits exist. Define $r_t^* \equiv \ln p_t^* - \ln p_{t-1}^*$ and $r_t \equiv \ln p_t - \ln p_{t-1}$ which are the continuously compounded, equilibrium and observed returns of the asset, respectively. With some simple algebra, it is easy to prove that the two returns are related as follows,

$$r_t = \begin{cases} \bar{c}_t & \text{if} \quad r_t^* + LO_{t-1} \geq \bar{c}_t \\ r_t^* + LO_{t-1} & \text{if} \quad \underline{c}_t < r_t^* + LO_{t-1} < \bar{c}_t \\ \underline{c}_t & \text{if} \quad r_t^* + LO_{t-1} \leq \underline{c}_t \end{cases} \tag{2}$$

where $\underline{c}_t \equiv \ln(1 - \frac{a}{p_{t-1}})$, $\bar{c}_t \equiv \ln(1 + \frac{a}{p_{t-1}})$ and $LO_{t-1} \equiv \ln(p_{t-1}^*/p_{t-1})$. Both $\underline{c}_t$ and $\bar{c}_t$ are contained in the econometrician's information set at time $t-1$. It might be worth pointing out that using continuously compounded return is *computationally* more attractive than using simple return in current circumstances.

To understand the structure (2), the term LO_{t-1} in it deserves a detailed discussion. From its definition, this term captures the unrealized return due to price limit move at time $t-1$. It was called a *leftover* term in Yang and Brorsen (1995). From a pure statistical point of view, if LO_{t-1} is always zero, the structure (2) is indeed the same as that in a two-limit Tobit model. Obviously, LO_{t-1} cannot be always zero in this case. Thus the censoring structure of a Tobit model is a misspecification for the asset *returns*. Essentially, this distinguishes my model from the ones used by Kodres (1993) and Morgan and Trevor (1997). A further interpretation of the leftover term is facilitated by the following concept.

Definition 2.1: A *price limit string* is a sequence of consecutive limited prices that immediately proceeds and follows an unlimited price, or a price without the imposition of price limits.[6]

At any non-limit move time t, LO_t is equal to zero by definition. Certainly, p_t^* does not belong to any price limit string. Now suppose $\{p_{t+1}^*, p_{t+2}^*, \cdots, p_{t+\tau}^*\}$ is a price limit string, which means that all p_{t+j}^* ($j = 1, 2, \cdots, \tau$) are unobserved and both p_t and $p_{t+\tau+1}$ are non-limit prices. The subscript τ is the length of the price limit string. It is then easy to derive

$$\begin{aligned} LO_{t+j} &= r_{t+j}^* + r_{t+j-1}^* + \cdots + r_{t+1}^* - (r_{t+j} + r_{t+j-1} + \cdots + r_{t+1}) \\ &= \sum_{i=1}^{j} (r_{t+i}^* - r_{t+i}) \end{aligned}$$

for all j ($1 \leq j \leq \tau$). This says that LO_{t+j} is the accumulated unrealized returns starting from the beginning of the price limit string to now (*i.e.*, $t+j$). It is thus understood that the structure (2) is a censoring structure for the asset returns, though it is different from the censoring structure of a Tobit model. If an upper (down) price limit is hit at time t, then $LO_t \geq (\leq) 0$. According to the above expression of LO_{t+j}, the following constraints for the equilibrium returns in the price limit string

$$\begin{cases} \sum_{i=1}^{j} r_{t+i}^* \geq \sum_{i=1}^{j} r_{t+i} & \text{if} \quad p_{t+j}^* \geq p_{t+j} \\ \sum_{i=1}^{j} r_{t+i}^* \leq \sum_{i=1}^{j} r_{t+i} & \text{if} \quad p_{t+j}^* \leq p_{t+j} \end{cases} \tag{3}$$

[6] If a sample starts with a limit move, the first price limit string then begins with the first price observation. Similarly, if a sample ends with a limit move, the last price limit string terminates with the last price observation.

must be true for any j ($1 \leq j \leq \tau$). In addition, the following equality constraint holds obviously

$$r_t^* + r_{t+1}^* + \cdots + r_{t+\tau+1}^* = r_t + r_{t+1} + \cdots + r_{t+\tau+1}. \tag{4}$$

These constraints imply that although the equilibrium returns in a price limit string are unobserved, they do stay in a constrained region. Furthermore, the inequalities in (3) and the equality in (4), plus $LO_t = 0$ if p_t is not a limited price, are equivalent to the censoring structure (2). **At this point, it can be easily proved that given an initial unlimited price p_1, the censoring structure (1) for the asset prices is equivalent to the censoring structure (2) for the asset returns.** One implication of the emphasized statement is that the censoring structure of a Tobit model for the asset returns is *not* equivalent to the censoring structure (1) for the asset prices.

To model the process of the equilibrium return r_t^*, the ARCH literature is followed. The ARCH model has been extensively studied since its introduction by Engle (1982). Bollerslev (1986) generalized it to GARCH models, which have proven attractive for the returns of most financial assets. The crux of these models is their ability to capture volatility clustering. Both Kodres (1993) and Morgan and Trevor (1997) followed this idea. For simplicity, this paper takes a parsimonious GARCH(1,1) model for the equilibrium return r_t^*, which is given by

$$\begin{aligned} r_t^* &= \mu + \varepsilon_t^* \sqrt{h_t^*}, \quad \epsilon_t^* | \mathcal{F}_{t-1} \sim \; GED \\ h_t^* &= \omega + \alpha h_{t-1}^* (\epsilon_{t-1}^*)^2 + \beta h_{t-1}^* \end{aligned} \tag{5}$$

where the innovation ε_t^* is orthogonal to all the available information at time t-1, $\mathcal{F}_{t-1}$; following Nelson (1991), $\varepsilon_t | \mathcal{F}_{t-1}$ is assumed to have the generalized error distribution (GED) with zero mean and unit variance;[7] the parameters governing the volatility function satisfy the typical restrictions: $\omega > 0$, $\alpha > 0$ and $\beta > 0$. The initial volatility h_1^* is assumed to be a known constant. The parameter μ is the one-period, continuously compounded return on the risk-free security. The density function of a GED random variable normalized to have a zero mean and unit variance is

$$f(z|\nu) = \frac{\nu \exp\left(-\frac{1}{2}\left|\frac{z}{\lambda}\right|^{\nu}\right)}{\lambda 2^{1+1/\nu} ?\,(1/\nu)} \qquad -\infty < z < \infty \tag{6}$$

where

$$\lambda = [2^{-2/\nu} ?\,(1/\nu) / ?\,(3/\nu)]^{1/2},$$

and $?(\cdot)$ is the gamma function. Parameter ν determines the tail-thickness of the density function and can take any value in the interval $(0, +\infty)$. The standard normal density function is the special case of $\nu = 2$. For $\nu > 2$, the density function has tails thicker than the normal density and for $\nu < 2$, the fat-tail phenomenon occurs. Use of the conditional GED in this study can be justified by two arguments. First, the evidence reported in the later sections of this paper shows that (observed) asset returns subject to price limits appear to have thin-tails rather than fat-tails and the conditional GED allows for the flexibility. Second, even though conditional equilibrium return can have fat tails and conditional t-distributions have been widely used by some researchers, Duan (1997b) pointed out that use of conditional t-distributions for modeling continuously compounded return implies an

[7]See Harvey (1981) and Box and Tiao (1973). In Box and Tiao (1973), this distribution is called the exponential power distribution.

unbounded expected simple asset return. Thus t-distributions are not a sensible choice in this situation. I call the model consisting of (2) and (5) a *censored-GARCH model* in order to distinguish it from a Tobit-GARCH model.

It should be noted that the choice of GARCH(1,1) parameterization of the equilibrium asset returns is by no means crucial for the applicability of the developed estimation method in this paper. For example, ARCH (Engle, 1982), GARCH (Bollerslev, 1986), EGARCH (Nelson, 1991), NGARCH (Engle and Ng, 1993), and GJR-GARCH (Glosten Jagannathan and Runkle, 1993), or more generally the Augmented GARCH(p,q) model (Duan, 1997b) can all be readily taken and the so-formed censored models can be uniformly dealt with by the estimation method proposed in this paper.

In order to proceed with a Bayesian approach to estimating the censored-GARCH model, a prior distribution of the model parameters must be specified. Let $\theta = \{\mu, \omega, \alpha, \beta, \nu\}$. The prior distribution of θ is specified in an improper fashion as

$$p(\theta) \propto \text{constant}$$

where[8]

$$\mu \in (-\infty, +\infty), \quad \omega \in (0, +\infty), \quad \alpha \in (0,1), \quad \beta \in (0,1) \text{ and } \quad \nu \in (0, +\infty).$$

No further difficulty will be added to the estimation procedure proposed in the later sections if the weak stationarity constraint $\alpha + \beta < 1$ is imposed. From a practical point of view, this improper prior can be replaced by the following proper prior,

$$p(\theta) = p(\mu)p(\omega)p(\alpha)p(\beta)p(\nu) \tag{7}$$

where

$$\mu \sim U(-\xi_1, \xi_2), \quad \omega \sim U(0, b), \quad \alpha \sim U(0,1), \quad \beta \sim U(0,1) \text{ and } \quad \nu \sim U(\eta_1, \eta_2)$$

and $U(\delta_1, \delta_2)$ denotes a uniform distribution in the interval (δ_1, δ_2); $\{\xi_1, \xi_2, \eta_1, \eta_2, b\}$ are viewed as hyperparameters, indexing the prior distribution. As long as the values of ξ_1, ξ_2, η_2 and b are large enough and that of η_1 is sufficiently close to 0, the above two prior distributions amounts to representing the same prior information. The posterior estimations in this paper are all based on the prior specification (7) with some variation in the choice of hyperparameters. It should also be indicated that the particular forms of the prior specification do not matter with respect to the posterior computations in this paper due to the flexibility of the method. Consequently, posterior estimations with more informative priors (including the small values of the hyperparameters) turn out to be trivial exercises. The choice of the hyperparameters is delayed to Sections 4 and 5.

3 The posterior approach

The likelihood function of the censored-GARCH model can be derived in a manner similar to that in Kodres (1993) or Lee (1997) or Wei (1997). Overall, it is analytically intractable due

[8] Although limited liability implies that the domain of μ is $[-1, +\infty)$, it is convenient and customary to use $(-\infty, +\infty)$ for the domain. So far no negative consequence has been reported with the practice in the empirical finance literature.

to the multiple dimensional integrals for the unobserved equilibrium returns, which prevents any analytical solution of posterior distribution and moments of the model parameters. This section develops a posterior estimation algorithm for the censored-GARCH model, based on the griddy Gibbs sampler-data augmentation algorithm (Ritter and Tanner, 1992).

3.1 The griddy Gibbs sampler-data augmentation

The Gibbs sampler-data augmentation algorithm is a well known sampling tool in econometrics. A brief review of this tool helps to introduce and understand the griddy Gibbs sampler-data augmentation algorithm. The basic idea can be explained in the simplest version of the tool. Suppose that the model parameters θ can be decomposed into two blocks, $\theta = (\theta_1, \theta_2)$, R is the vector of total observed returns r_t , and r^* the vector of total unobserved equilibrium returns r_t^* due to price limit moves at time t and/or $t-1$. If the complete conditional distributions

$$\theta_1|\{\theta_2, r^*, R\}, \qquad \theta_2|\{\theta_1, r^*, R\} \quad \text{and} \quad r^*|\{\theta_1, \theta_2, R\} \tag{8}$$

are all in standard forms (for instance, normal and gamma distributions) from which random numbers could be easily sampled, then the Gibbs sampler-data augmentation algorithm is to iteratively draw from these conditionals. As the number of draws grows large, the draws so obtained converge in distribution to that of the joint posterior distribution of the parameters θ and the unobserved returns r^*. (For references, see Gelfand and Smith (1990) and Tanner and Wong (1987).) What if the conditional distributions are in non-standard forms? A (more) *numerical* version of the Gibbs sampler-data augmentation algorithm was developed by Ritter and Tanner (1992).[9] One can evaluate each conditional distribution over a grid of points and then generate a draw from the simulated conditional distribution by inverting it at a value sampled from the uniform distribution in $(0, 1)$. A detailed description of the procedure is given in the Appendix. The implementation of this procedure requires that each conditional distribution be one-dimensional since it is the case in which the above-mentioned numerical evaluation can be conveniently proceeded. The posterior estimation is then straightforward with the simulated posterior draws (see a detailed discussion by Bauwens and Lubrano (1998)). The convergence of each posterior Markov chain can be easily checked by using the visual inspection of CUMSUM statistics proposed by Yu and Mykland (1994). A standardized version of the statistic can be written as, with N draws of a Monte Carlo Markov chain $\theta^{(n)}$,

$$CS_t = (\frac{1}{t}\sum_{n=1}^{t}\theta^{(n)} - \mu_\theta)/\sigma_\theta, \quad \text{for } t = 50, 100, 150, ..., N$$

where μ_θ and σ_θ are the empirical mean and standard deviation of the N draws. If the MCMC chain converges, then the plot of CS_t against time t should converge smoothly to zero. On the other hand, a long and regular excursion plot of CS_t indicates the absence of convergence of the chain. Bauwens and Lubrano (1998) refined the idea by introducing an $\epsilon-$*band* for CS_t. If CS_t remains in the $\epsilon-$*band (around zero)* for all t larger than $K(\epsilon)$, then $\theta^{(n)}$ has converged after $K(\epsilon)$ draws for the estimation of the posterior mean with a relative error of $100 \times \epsilon$ percent.

[9] They did not consider the data augmentation, but it adds no more difficulty if the data augmentation is incorporated into their method.

3.2 Conditional distributions

This subsection focuses on the derivation of the complete conditional distributions of the parameter θ and the latent returns r^*. I begin with the "latent likelihood function" of the model, *i.e.*, the sampling distribution of the total equilibrium returns R^* (R^* can be viewed as the union of R and r^*). Because R^* is not fully observable, I use the word "latent" to capture the essential idea. The latent likelihood can be easily written as

$$L^*(\mu, \omega, \alpha, \beta | R^*) = \prod_{t=1}^{T} \frac{\nu \exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\sqrt{h_t^*}\lambda 2^{1+1/\nu} ?\,(1/\nu)} \tag{9}$$

where $h_t^* = \omega + \alpha h_{t-1}^*(\epsilon_{t-1}^*)^2 + \beta h_{t-1}^*$. The "latent posterior distribution" is defined accordingly, *i.e.*, the latent likelihood function multiplied by the model prior (see (7)). The conditional posterior distributions of the parameters have the following density kernels

$$\begin{aligned}
\mu|\omega, \alpha, \beta, \nu, R^* &\sim \prod_{t=1}^{T} \frac{\exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\sqrt{h_t^*}} \quad \xi_1 < \mu < \xi_2, \\
\omega|\mu, \alpha, \beta, \nu, R^* &\sim \prod_{t=1}^{T} \frac{\exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\sqrt{h_t^*}} \quad 0 < \omega < b, \\
\alpha|\mu, \omega, \beta, \nu, R^* &\sim \prod_{t=1}^{T} \frac{\exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\sqrt{h_t^*}} \quad 0 \le \alpha < 1, \\
\beta|\mu, \omega, \alpha, \nu, R^* &\sim \prod_{t=1}^{T} \frac{\exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\sqrt{h_t^*}} \quad 0 \le \beta < 1, \\
\nu|\mu, \omega, \alpha, \beta, R^* &\sim \prod_{t=1}^{T} \frac{\nu \exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_t^*}\right|^{\nu}\right]}{\lambda 2^{1+1/\nu} ?\,(1/\nu)} \quad 0 < \nu < +\infty.
\end{aligned} \tag{10}$$

The conditional distributions seem to possess the same form, but they are viewed rather differently because each of the variables conditions on all the others. Obviously, they are all in non-standard forms, which motivates the use of the griddy method.

Now consider the conditional distribution of the latent data as required in the data augmentation step discussed above. A corresponding concept to price limit string is now defined for returns.

Definition 3.1: A *latent* (*return*) *string* is a sequence of consecutive unobserved returns that follows and proceeds immediately an uncensored return.

Unlike in a dynamic Tobit model, the unobserved returns conditioning on the model parameters and all observations are *not* independent across different latent strings. This is easily seen from the kernel of the conditional density of the latent returns $r_{t+1}^*, \cdots, r_{t+n_t}^* | \{\theta, R^* - \{r_{t+1}^*, \cdots, r_{t+n_t}^*\}\}$ in a latent string $\{r_{t+1}^*, \cdots, r_{t+n_t}^*\}$

$$\frac{\exp\left[-\frac{1}{2}\left|(r_{t+n_t+1} - \mu)/\lambda\sqrt{h_{t+n_t+1}^*}\right|^{\nu}\right]}{\sqrt{h_{t+n_t+1}^*}} \prod_{j=1}^{n_t} \frac{\exp\left[-\frac{1}{2}\left|(r_t^* - \mu)/\lambda\sqrt{h_{t+j}^*}\right|^{\nu}\right]}{\sqrt{h_{t+j}^*}} \tag{11}$$

where n_t is the length of the latent string. The first term in this density kernel shows up because $h_{t+n_t+1}^*$ is a function of the latent return $r_{t+n_t}^*$ according to the model specification

(5). The unobserved returns in this latent string are linked to those in the past latent strings through the volatility function h_t^*. If the volatility function is determined by observed returns rather than unobserved returns, the latent returns in this latent string conditioning on the model parameters and all observables are *conditionally* independent of the latent returns in other latent strings. In this case, the joint density of all latent returns can be written as the product of the density of the latent returns in a latent string over all such strings. This is analogous to the case of a dynamic Tobit model as discussed in Wei (1997). As a result, the data augmentation step becomes simpler.

Notice that the censoring structure (2) implies the constraints (3) and (4) for the latent returns. Thus the distribution of the latent returns in a latent string, conditioning on all the model parameters and other latent returns and observables, turns out to be (11) subject to the constraints (3) and (4). It is interesting to see that a change-of-variable technique can solve the sampling problem nicely. Set

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} r_{t+1}^* \\ r_{t+2}^* \\ \vdots \\ r_{t+n_t}^* \end{bmatrix} = \begin{bmatrix} x_{t+1} \\ x_{t+2} \\ \vdots \\ x_{t+n_t} \end{bmatrix} \tag{12}$$

where the coefficient matrix is of a lower triangle. The reason for making the transformation is that the transformed variables have simple and neat forms of constraints, and are easier to be sampled. According to the change-of-variable technique, it is easily confirmed that the density kernel of the transformed random variables $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t}$ has the following form

$$\frac{\exp\left[-\frac{1}{2}\left|(r_{t+n_t+1}-\mu)/\lambda\sqrt{h_{t+n_t+1}}\right|^{\nu}\right]}{\sqrt{h_{t+n_t+1}}} \prod_{j=1}^{n_t} \frac{\exp\left[-\frac{1}{2}\left|(x_{t+j}-x_{t+j-1}-\mu)/\lambda\sqrt{h_{t+j}}\right|^{\nu}\right]}{\sqrt{h_{t+j}}} \tag{13}$$

where the determinant of the Jacobian of the inverse transformation of (12) equals 1,

$$\begin{aligned} h_{t+1} &= \omega + \alpha(r_t - \mu)^2 + \beta h_t^* \\ h_{t+2} &= \omega + \alpha(x_{t+1} - \mu)^2 + \beta h_{t+1} \\ h_{t+j} &= \omega + \alpha(x_{t+j-1} - x_{t+j-2} - \mu)^2 + \beta h_{t+j-1} \quad j = 3, 4, \cdots, n_t + 1, \end{aligned}$$

$x_{t+0} \equiv 0$ (for notational convenience), the constrains (3) and (4) are easily transformed into

$$\begin{cases} x_{t+j} \geq \sum_{i=1}^{j} r_{t+i} & \text{if} \quad p_{t+j}^* \geq p_{t+j} \\ x_{t+j} \leq \sum_{i=1}^{j} r_{t+i} & \text{if} \quad p_{t+j}^* \leq p_{t+j} \end{cases} \tag{14}$$

for $1 \leq j < n_t$ and

$$x_{t+n_t} = \sum_{i=1}^{n_t} r_{t+i}^*. \tag{15}$$

The equality constraint allows one-dimensional reduction of the random vector to be sampled. Set

$$A_{t+j} = \{x_{t+j} | x_{t+j} \geq \sum_{i=1}^{j} r_{t+i} \text{ if } p_{t+j}^* \geq p_{t+j} \text{ and } x_{t+j} \leq \sum_{i=1}^{j} r_{t+i} \text{ if } p_{t+j}^* \leq p_{t+j}\}$$

where $j = 1, 2, \cdots, n_t - 1$. Then the sampling problem becomes to draw $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t-1}$ from the following truncated distribution

$$\frac{\exp\left[-\frac{1}{2}\left|(r_{t+n_t+1} - \mu)/\lambda\sqrt{h_{t+n_t+1}}\right|^{\nu}\right]}{\sqrt{h_{t+n_t+1}}} \prod_{j=1}^{n_t-1} \frac{\exp\left[-\frac{1}{2}\left|(x_{t+j} - x_{t+j-1} - \mu)/\lambda\sqrt{h_{t+j}}\right|^{\nu}\right]}{\sqrt{h_{t+j}}} I_{\{x_{t+j} \in A_{t+j}\}} \tag{16}$$

where I is an indicator function. After drawing $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t-1}$ from the kernel (16), the draws of $r^*_{t+1}, \cdots, r^*_{t+n_t}$ can be easily obtained by using the inverse transformation of (12). The above discussion is summarized as follows.

Summary: For the censored-GARCH model, the equilibrium returns in a latent string $r^*_{t+1}, \cdots, r^*_{t+n_t}$ can be sampled as follows, for the data augmentation step,

- *step 1*: transform the equilibrium returns into $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t}$ by using (12).
- *step 2*: sample the variates $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t-1}$ from the truncated density kernel (16).
- *step 3:* transform the draws of $x_{t+1}, x_{t+2}, \cdots, x_{t+n_t-1}$ into the draws of $r^*_{t+1}, \cdots, r^*_{t+n_t}$ according to (4) and (12).

Because (16) is still in a non-standard form, the vector $\{x_{t+1}, x_{t+2}, \cdots, x_{t+n_t-1}\}$ needs to be further partitioned into $n_t - 1$ univariate variables and the kernel of the distribution of each univariate variable x_{t+j} is

$$\frac{\exp\left[-\frac{1}{2}\left|(r_{t+j+1} - \mu)/\lambda\sqrt{h_{t+j+1}}\right|^{\nu} - \frac{1}{2}\left|(x_{t+j} - x_{t+j-1} - \mu)/\lambda\sqrt{h_{t+j}}\right|^{\nu}\right]}{\sqrt{h_{t+j}h_{t+j+1}}} I_{\{x_{t+j} \in A_{t+j}\}} \tag{17}$$

where h_{t+j} is defined above. This non-standard density kernel again motivates the use of the griddy method.

So far, all the full conditional density kernels of the model parameters and latent data have been derived. The posterior output can be now obtained by applying the griddy Gibbs sampler-data augmentation as given before.

4 An application to Treasury bill futures

This section considers an application of the proposed model and estimation method. The data contain the prices of the 3-month US Treasury bill (T-bill) futures and price limit dates.[10] The background and description of the data are briefly discussed, a preliminary analysis is conducted, for the sake of comparison, two other models, namely a GARCH (1,1) model and a Tobit-GARCH model, are also introduced, and then the posterior results of all the models are reported and compared.

4.1 Background and data

The contract of the 3-month T-bill futures was first introduced in January, 1976, at the International Monetary Market (IMM), a division of the Chicago Mercantile Exchange (CME). It has been playing a major role in hedging short-run interest rate risks. So far, it is the most

[10]The data are provided by I.G. Morgan.

heavily traded instrument among all interest rate futures. The sample spans from October 1, 1979 to October 29, 1982, which represents a special episode in the history of the Federal Reserve System (the Fed). It is well-known that during the period, the Fed adopted a monetary operating procedure that was aimed at combating a non-tolerable inflation rate. This so-called monetarist-advocating strategy is to control money supply so that interest rates were allowed to adjust more freely. The direct impact of the Fed's strategy on the T-bill futures was the high volatility in the futures market during the whole period (see Figure 1).

As documented in International Monetary Market Yearbook (1983, p52), during the sampled period, the daily price limits in the futures market were regulated at the levels of 50 basis points above or below the previous day's settlement price before June 19, 1980 and then raised to those of 60 basis points. The price limit moves of the *first deferred contract* of the T-bill futures are 57 days. Figure 2 plots the daily prices of the futures contracts and their reached limits (with the symbol $\times$) against time. The ratio of price limit days to total observations is 7.3%. Of 57 limit days, 18 are two-day consecutive limit moves in the same direction and 2 occurred in the opposite direction.

4.2 Preliminary analysis

The time series plot of the T-bill futures returns is displayed in Figure 1 in which volatility clustering of the returns is easily confirmed. The sample statistics of the data are reported in Table 1. It is noted that the excess kurtosis of the observed returns is negative ($-.39$), which seems to suggest a contradiction with the well-known fat-tailed phenomenon. However, if price limits are taken into account, I argue that this contradiction can be easily reconciled. First, price limits prohibit extreme returns by restricting large movement of prices. As a result, this reduces the excess kurtosis of the observed returns, meaning that the observed returns can be thin-tailed even if the equilibrium returns are fat-tailed.

Next, the appearing contradiction can be explained with an examination of the (unconditional) density of the observed returns. This density is unknown, but can be easily estimated in a simple non-parametric fashion. A kernel estimation method is adopted and the selected kernel is given by

$$K(u) = \frac{15}{16}(1-u^2)^2 I(|u| \leq 1). \tag{18}$$

This is the typical Biweight kernel. The bandwidth is taken as $2.78T^{-\frac{1}{5}}\hat{\sigma}$ where T is the sample size and $\hat{\sigma}$ is the sample standard deviation of the observed return r_t (For a reference, see Silverman (1986)). The bandwidth is selected based on the widely-used criterion of minimizing the approximate mean integrated square error, under the assumption that the "true" distribution of the observed returns r_t is normal (see Silverman (1986, p. 40)). The density estimate (solid curve) as well as a simulated normal density (dotted curve) are depicted in Figure 3. The normal density serves as a reference point, with the same mean and standard deviation as those of the observed returns. At first glance, it is surprising that two *humps* appear on the tails of the estimated density. It is found that they are quite robust with respect to alternative choices of bandwidth, as long as the bandwidth is not too large. What generates the humps? The answer is price limits, simply because price limits affect (both positive and negative) large returns which are located in the two tails of the estimated density. Now, it is easy to understand that the tail behavior of the estimated density of the observed returns is consistent with the negative excess kurtosis of the return sample. Furthermore, a comparison between the two densities in Figure 3 suggests that the equilibrium returns of the futures contracts have a fat-tailed unconditional distribution because these returns are not restricted by price limits and have the same behavior as the observed returns when tails are ignored. At this point, hopefully it is convinced that price limits should be neither ignored nor deleted.

4.3 Two other models

It may be interesting to compare the estimation results of the censored-GARCH model with those of two relevant models. For the purposes, first consider a GARCH(1,1) model for the observed return r_t, which represents the ignorance of price limits,

$$\begin{aligned} r_t &= \mu + \varepsilon_t \sqrt{h_t}, \quad \epsilon_t | \mathcal{F}_{t-1} \sim \ GED \\ h_t &= \omega + \alpha h_{t-1} (\epsilon_{t-1})^2 + \beta h_{t-1} \end{aligned} \tag{19}$$

where all the notations here are the same as in Section 2, $\epsilon_t | \mathcal{F}_{t-1}$ has zero mean and unit variance and the typical restrictions on the volatility parameters are satisfied: $\omega > 0, \alpha > 0$ and $\beta > 0$. It is noted that this model is the same as the model (5) except that here r_t is naively treated as an equilibrium return. If this naive treatment is inappropriate, then a significant difference between the posterior results of this model and the censored-GARCH model should be observed.

Next consider Kodres'(1993) Tobit-GARCH model, with the replacement of a (conditional) normal error term with a conditional GED error term, which can be generally written as

$$r_t = \begin{cases} \bar{c}_t & \text{if} \quad r_t^* \geq \bar{c}_t \\ r_t^* & \text{if} \quad \underline{c}_t < r_t^* < \bar{c}_t \\ \underline{c}_t & \text{if} \quad r_t^* \leq \underline{c}_t \end{cases} \tag{20}$$

and

$$\begin{aligned} r_t^* &= \mu + \varepsilon_t^* \sqrt{h_t^*}, \quad \epsilon_t^* | \mathcal{F}_{t-1} \sim \ GED \\ h_t^* &= \omega + \alpha h_{t-1}^* (\epsilon_{t-1}^*)^2 + \beta h_{t-1}^* \end{aligned} \tag{21}$$

where again the notations here are the same as in Section 2, $\epsilon_t^* | \mathcal{F}_{t-1}$ has zero mean and unit variance and the typical restrictions on the volatility parameters are satisfied: $\omega > 0, \alpha > 0$ and $\beta > 0$. As discussed before, this model is a misspecification on the link between the observed return r_t and its equilibrium counterpart r_t^* in certain time points. For instance, the equality constraints on consecutive unobserved equilibrium returns (see (4)) are overlooked in this specification. As a result, this increases the dimension of the unobserved equilibrium returns in such a sample. On any single limit day t, Morgan and Trevor (1997) converted the equilibrium price p_t^* into two unobserved equilibrium returns r_t^* and r_{t+1}^*, leaving the equality constraint on the two returns unconsidered. On any two consecutive limit days, say t and $t+1$, the two equilibrium prices p_t^* and p_{t+1}^* were converted into three unobserved equilibrium returns r_t^*, r_{t+1}^* and r_{t+2}^*, with the ignorance of the equality constraint on them. If the two day limits are in the same direction, r_{t+1}^* was treated as a missing variable since there is no information to constraint it in this treatment. The loss of information due to a mistreatment of price limits in a Tobit-GARCH model may result in an estimation distortion of the model, or at least an efficiency loss of the estimation.

4.4 Posterior results

This subsection reports and compares the posterior results of the three models, two of which are given in the last subsection, and another of which is the censored-GARCH model, with the T-bill futures data. The same prior specification (7) is applied to all the three models. The hyperparameters are: $\xi_1 = \xi_2 = .12$, $b = .35$, $\eta_1 = 0.1$ and $\eta_2 = 4.5$. The choice of the particular values for hyperparameters is not attempted to inject much prior information into the analysis. Instead, it is made mainly based on computational efficiency, meaning that further extension of the supports of these parameters would not significantly alter the posterior results except that more computing time is involved.

Table 2 and Figure 4 present the posterior results. Two comments can be drawn from them. First, both the censored-GARCH model and the Tobit-GARCH model suggest a (conditional) fat-tailed distribution ($\nu < 2$) for the underlying equilibrium returns, which is consistent with the fat-tailed phenomenon, while the GARCH model implies a (conditional) thin-tailed distribution. This simply concludes that price limits should *not* be ignored because of the substantial difference in their data distributions. Second, the major difference between the estimates of the censored-GARCH model and those of the Tobit-GARCH model is reflected in the parameters of their volatility functions. Although it may not be easy to judge how much the parameter estimates of the Tobit-GARCH model are distorted due to the misspecification of the model, we can see that the variances of the estimated volatility parameters are larger for the Tobit-GARCH model than for the censored-GARCH model. This can be further confirmed in Figure 4 in which the posterior histograms of the parameters for the three models are plotted. This is not surpring because the Tobit-GARCH model introduces some additional uncertainty to the model estimation.

For the estimation of these models, I rescale the return data, multiplying 10^3 by them, so that ω needs to be divided by 10^6, μ divided by 10^3. The modified version of the CUMSUM evolution of the Monte Carlo estimates of the posterior means of the parameter for the censored-GARCH model is displayed in Figure 5 (see the discussion in Section 3.1). The error band ϵ is chosen as .1. The plots suggest that to ensure the Markov chains of the sampled parameters stay in the $\epsilon - band$, or converge in terms of the criterion discussed in Section 3.1, the first 6000 draws must be dropped. The high cost of dropping so many initial draws is due to the high correlation between the posterior parameters ω and β. The Gibbs sampler-data augmentation retains next 4000 draws. Additional draws have been also tried, but could not significantly improve the posterior results. I have also examined the modified CUMSUM plots for all the posterior latent returns, but do not report them here to save space. The program is coded in GAUSS and implemented in a Compucon Intel 430Hx Pentium 200 PC. The CPU time consumed in the computation is about 12 hours with $10,000$ draws for the censored-GARCH model.

5 A simulation example

This section offers a simulation example to ground the findings obtained in the last section. For the purposes, this example is designed to share certain major characteristics with the real data studied in the last section so that it is easy to compare the results here with those there.

Example 5.1: Let r_t^* be an equilibrium daily return series generated from the following

DGP,

$$\begin{aligned} r_t^* &= 0 + \varepsilon_t^* \sqrt{h_t^*}, \quad \epsilon_t^* | \mathcal{F}_{t-1} \sim \ N(0,1) \\ h_t^* &= .05/10^6 + .05 h_{t-1}^* (\epsilon_{t-1}^*)^2 + .90 h_{t-1}^* \end{aligned} \tag{22}$$

with the sample size 1000. The daily equilibrium price p_t^* is initialized at $p_1^* = 90$ and constructed as follows,

$$p_t^* = \exp(\ln p_{t-1}^* + r_t^*)$$

where r_t^* resembles a daily return used in the last section. The observed daily price series is generated by

$$p_t = \begin{cases} p_{t-1} + .17 & \text{if} \quad p_t^* \geq p_{t-1} + .17 \\ p_t^* & \text{if} \quad p_{t-1} - .17 < p_t^* < p_{t-1} + .17 \\ p_{t-1} - .17 & \text{if} \quad p_t^* \leq p_{t-1} - .17 \end{cases} \tag{23}$$

where .17 is used to define daily price limits and is so selected to achieve roughly the same number of limit moves as that in the application of the last section. The initial 200 simulated prices are thrown away to reduce their likely effect and the rest of the simulated prices are kept for this analysis. Thus the total number of price observations is 800. The number of simulated price limit moves is 58, of which 3 pairs are two-day consecutive limit moves and all others are single day limit moves. Recall that the observed and equilibrium daily returns were defined in Section 2. To be comparable with the application in the last section, both r_t and r_t^* are rescaled by multiplying 10^3.

Sample statistics of the simulated (observed and equilibrium) returns are reported in Table 3. In this example, the excess kurtosis of the unconditional observed return r_t is also *negative* though the excess kurtosis of the equilibrium return r_t^* is still positive. Obviously, the negative kurtosis can only be caused by price limits in this circumstance. The interpretation of the negative excess kurtosis in the last section is supported. Either a normal or Student-t version of GARCH model cannot fit the fourth moment of r_t since such models imply a positive excess kurtosis of r_t. The message behind this observation is that ignoring or deleting price limits is indeed inappropriate.

The (unconditional) density estimation of the simulated return r_t is conducted in the same manner as that in the last section. For a reference, the (unconditional) density estimation of r_t^* (the data r_t^* are available because they are simulated) is also presented. The estimated two densities are displayed in Figure 6, with a simulated normal density. The normal density is designed with the same mean and standard deviation as those of the sample r_t. Similarly to what we observed in the last section, two humps appear on the tails of the estimated density of r_t and they are quite robust with respect to alternative choices of bandwidth. Clearly, they are induced by price limits because no hump arises in the estimated density of r_t^*. This confirms the major finding in the preliminary analysis of the last section and suggests that price limits should be neither ignored nor deleted. In general, it is easily understood that price limits may not always induce negative excess kurtosises of such samples, which should depend on the relative number of limited prices. As the number increases, the excess kurtosises of the samples would decrease.

Following what have been done in the last section, I also estimate the three models there with the simulated returns r_t. Table 4 and Figure 7 report the posterior results. It is impressive that the results from the censored-GARCH model are better than those from both the Tobit-GARCH model and the GARCH model. This is not surprising because ignoring price limits makes the fourth moment of the data and the humps on the tails of the data hard to fit, and the Tobit-GARCH model misspecifies the observed return process. Interestingly, the posterior estimation results and distributions here overwhelmingly confirm those in the last section. For example, posterior draws of tail-thickness parameter ν have a mean fairly close to its true value 2 for the censored-GARCH and Tobit-GARCH models, but not for the GARCH model. In addition, the information loss due to the Tobit-GARCH model reduces the estimation efficiency of the model.

The implementation details of estimations of the three models are exactly the same as those in the last section. They are not reported here, because with the true data generating process known, it is easy to see the adequacy of the proposed method, and the accuracy of the posterior results.

6 Conclusion

This paper has formulated a censored-GARCH model to describe the return process of the assets subject to daily price limits. This model differs from a Tobit-GARCH model as posted by Kodres (1993) and further studied by Morgan and Trevor (1997) in at least one major aspect. While the censored-GARCH model implies a set of linear constraints on the unobserved equilibrium returns required by price limits, a Tobit-GARCH model is not able to fully capture these constraints and introduces some unnecessary uncertainty to the model estimation.

Furthermore, this paper has offered a simple and practical Bayesian estimation technique for the censored-GARCH model, which is built on the griddy Gibbs sampler-data augmentation algorithm (cf. Ritter and Tanner (1992)). Sampling from consecutive unobserved equilibrium returns consists of the key part of this developed estimation technique. I demonstrated that this sampling procedure can be nicely and easily implemented by using a simple change-of-variable technique combined with the griddy method.

Several major advantages of the proposed estimation method are worth being summarized. First, it allows for flexibility on both prior and model specifications. Second, it provides a general and simple sampling procedure to draw variates from a distribution with a set of linear constraints (A truncated distribution is a special case). Third, it can be easily generalized to estimate other censored and/or non-linear regression models. Lastly, it is straightforward to code and implement in almost all routinely-used statistical software.

An application study and a simulation example show the worthiness of the development of the new model. A few main results have been derived. First, price limits can result in negative kurtosis of the sample of the observed returns, though the distribution of underlying equilibrium returns may still be fat-tailed. Second, price limits can distort the tail behavior of the distribution of the observed returns, which may further explain why the sample of the observed returns is thin-tailed (*i.e.*, negative kurtosis). Clearly, it is hard, if not impossible, to fit these important features of such data if price limits are ignored or deleted. Consequently, this paper calls for a serious consideration of taking account of price limits in dealing with such samples. Third, a Tobit-GARCH model would result in distortions because it is a misspecification to the observed return process of the assets subject to daily price limits. In particular, some additional uncertainty due to the model reduces the estimation efficiency of the model. Therefore, the censored-GARCH model is strongly recommended for future studies. Finally, both the real application and the simulation example delivered non-normal posterior distributions of the parameters of the volatility functions, which is the strength of the Bayesian estimation method in finite samples.

In an on-going research, I am making a comparison study of the performances of the proposed estimation method, the possible extensions of the maximum simulated likelihood method used, for example, by Lee (1997), and the EM method. I am also investigating the financial and economic implications of price limits by using some variations of the model proposed in this paper. The flexibility of the developed method should make this investigation easy and convenient.

Table 1: Sample statistics of T-bill futures returns
(rescaled by 1000) October 1, 1979 – October 29,1982

	Sample Size	Mean	Std. Dev.	Ske.	Excess Kurt.	# of price limits
r_t	777	-.0037	0.75	.0416	-.39	57

Table 2: Estimation results of T-bill futures returns

		Parameter				
Model ↓		μ	ω	α	β	ν
GARCH	post. mean	-.01	.06	.06	.83	2.50
	post. std. dev.	(.03)	(.06)	(.03)	(.12)	(.32)
Tobit-GARCH	post. mean	.0	.05	.05	.87	1.63
	post. std. dev.	(.03)	(.05)	(.03)	(.08)	(.18)
Censored-GARCH	post. mean	.0	.03	.04	.91	1.62
	post. std. dev.	(.03)	(.03)	(.02)	(.06)	(.18)

Note: ω is multiplied by 10^6 and μ by 10^3.

Table 3: Sample statistics of Example 5.1

	Sample Size	Mean	Std. Dev.	Ske.	Excess Kurt.	# of Limited Prices
r_t^*	799	.027	1.05	.001	.29	
r_t	799	.027	.98	-.061	-.55	58

Table 4: Estimation results of Example 5.1

		Parameter				
		μ	ω	α	β	ν
Model ↓		0	.05	.05	.90	2.0
GARCH	post. mean	.02	.18	.06	.76	3.04
	post. std. dev.	(.03)	(.12)	(.03)	(.12)	(.46)
Tobit-GARCH	post. mean	.03	.14	.06	.81	1.80
	post. std. dev.	(.04)	(.09)	(.03)	(.09)	(.19)
Censored-GARCH	post. mean	.03	.11	.05	.85	1.82
	post. std. dev.	(.04)	(.08)	(.02)	(.08)	(.19)

Note: ω is multiplied by 10^6 and μ by 10^3.

Figure 1: Return of T−bill futures price
(rescaled by 1000)

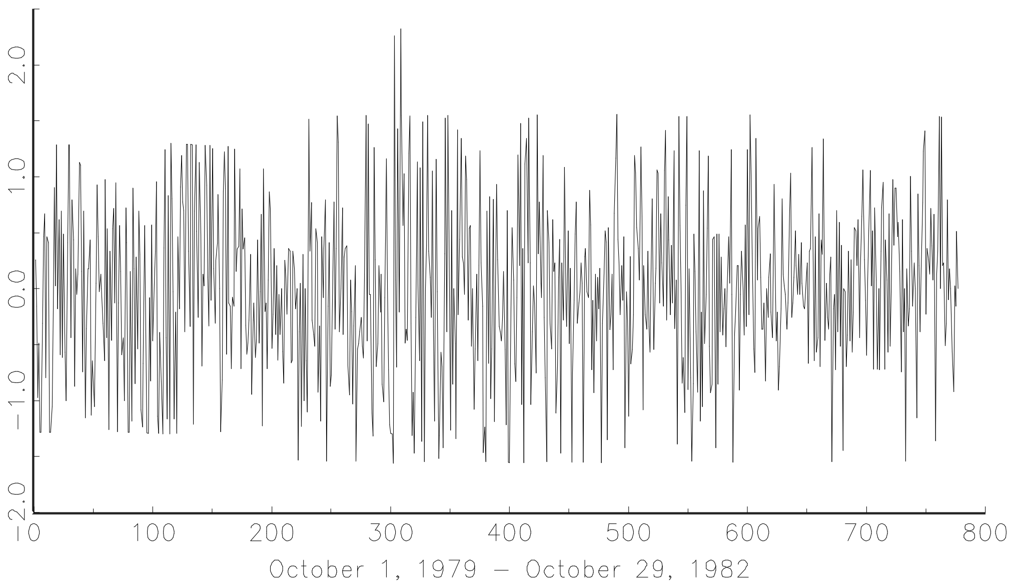


Figure 2: T−bill futures price with price limits

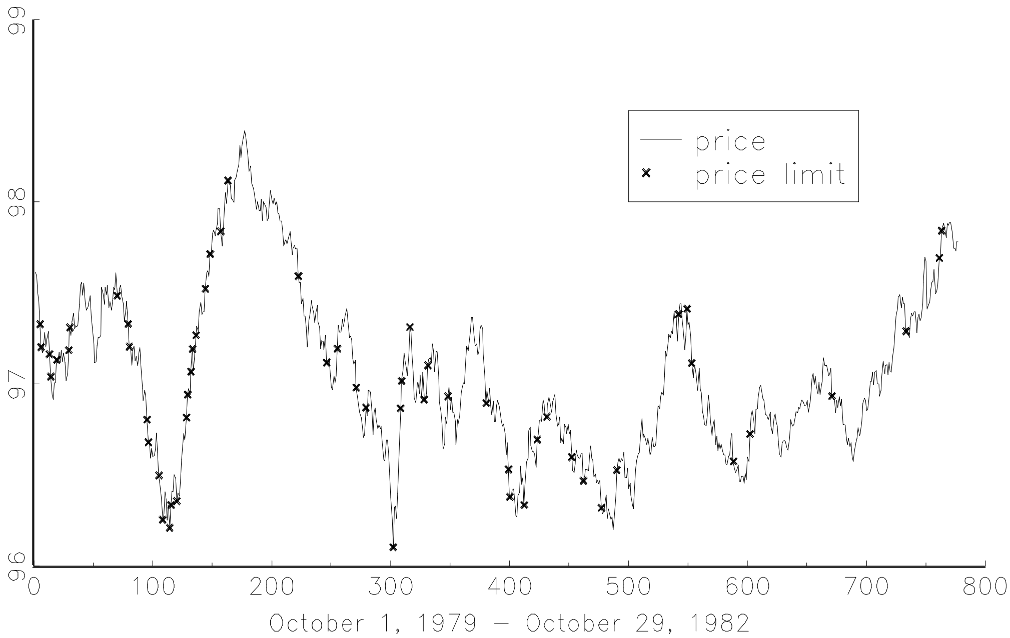


Figure 3: Nonparametric density estimation of the return of T−bill futures (rescaled by 1000)

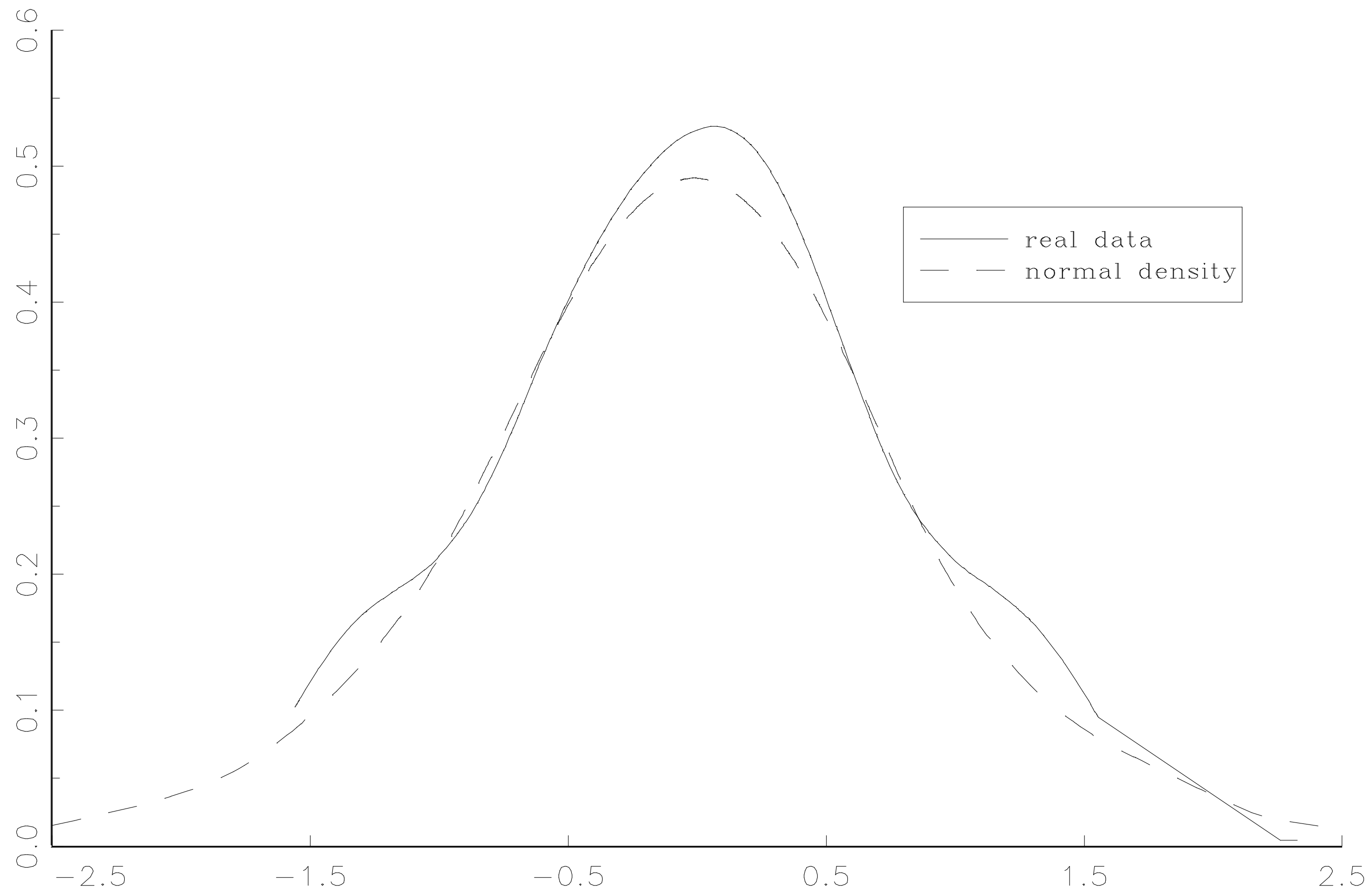


Figure 4: Histogram comparison of posterior parameters

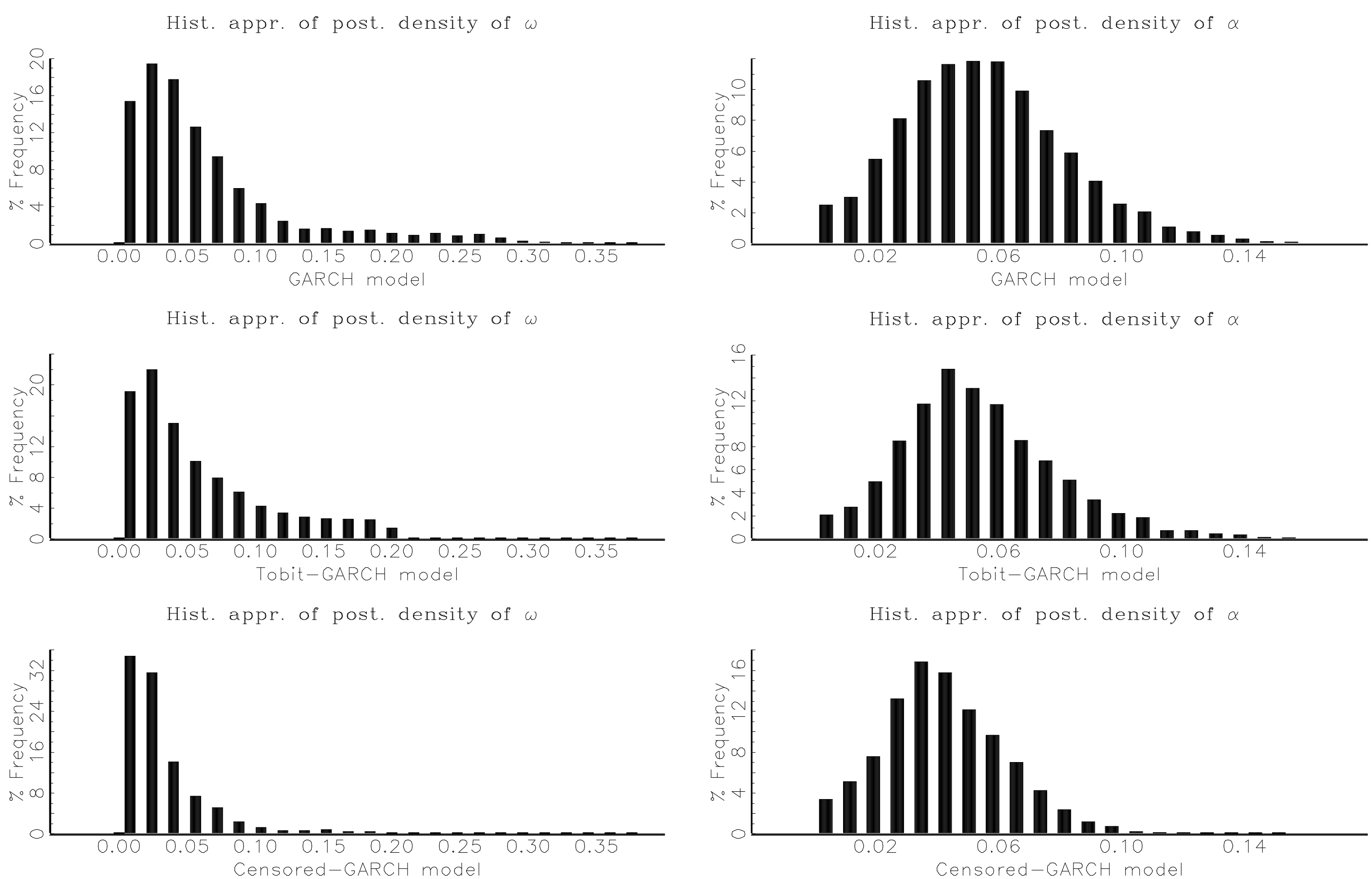


Figure 4: Histogram comparison of posterior parameters
(continued)

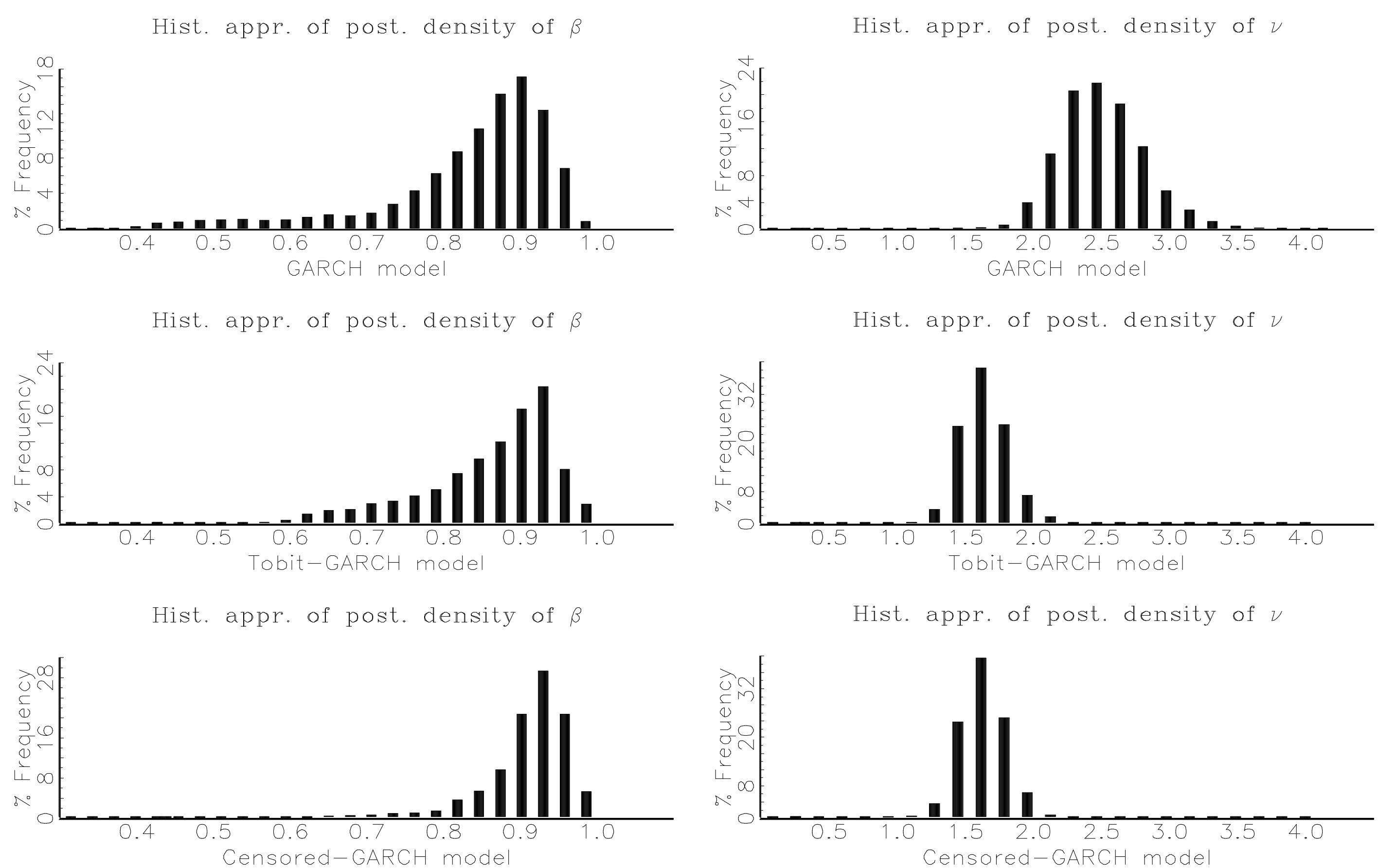


Figure 5 CUMSUM plots of posterior means estimates
(Censored-GARCH model with the return of T-bill futures)

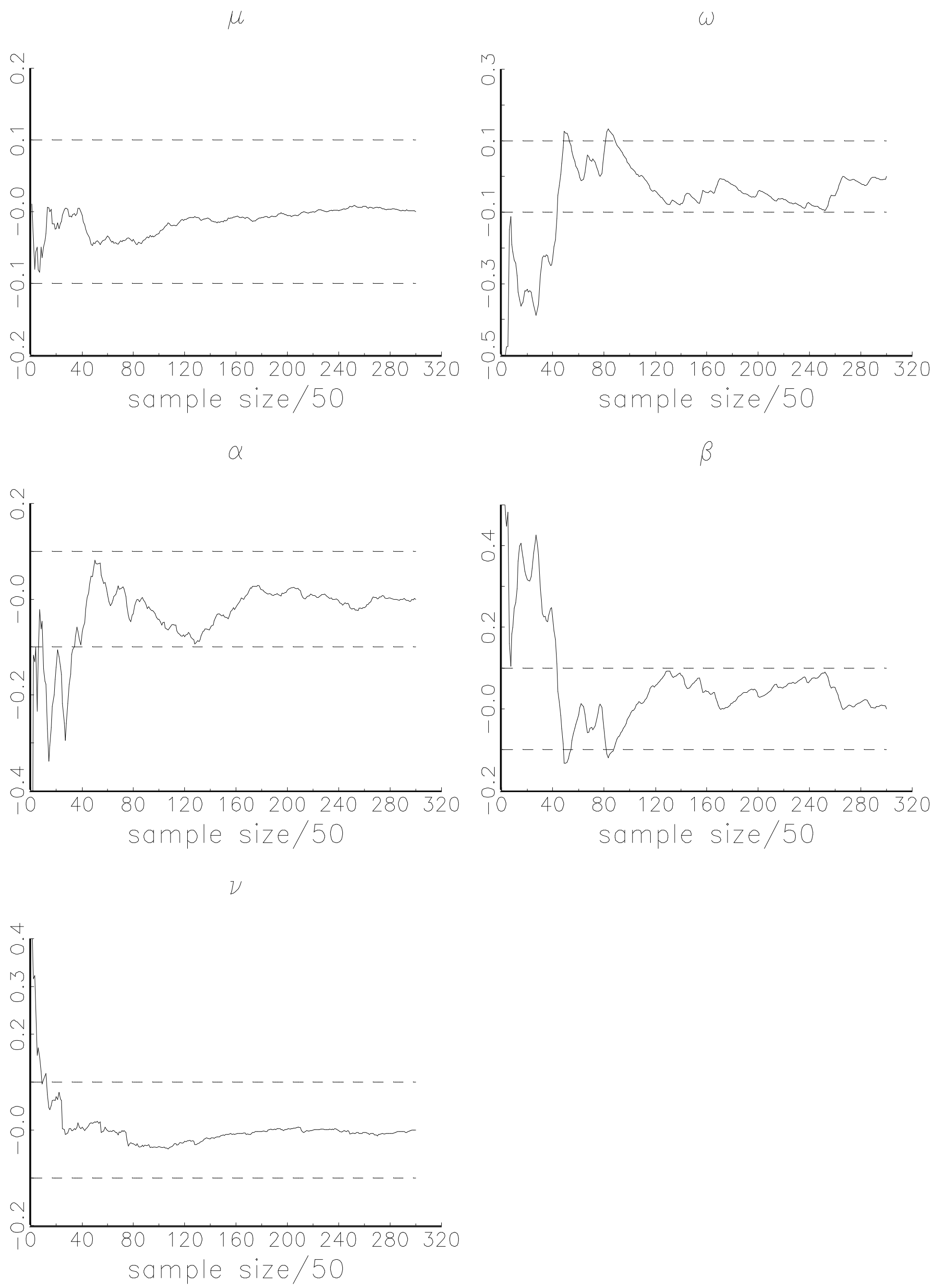


Figure 6: Nonparametric density estimation of simulated returns (rescaled by 1000)

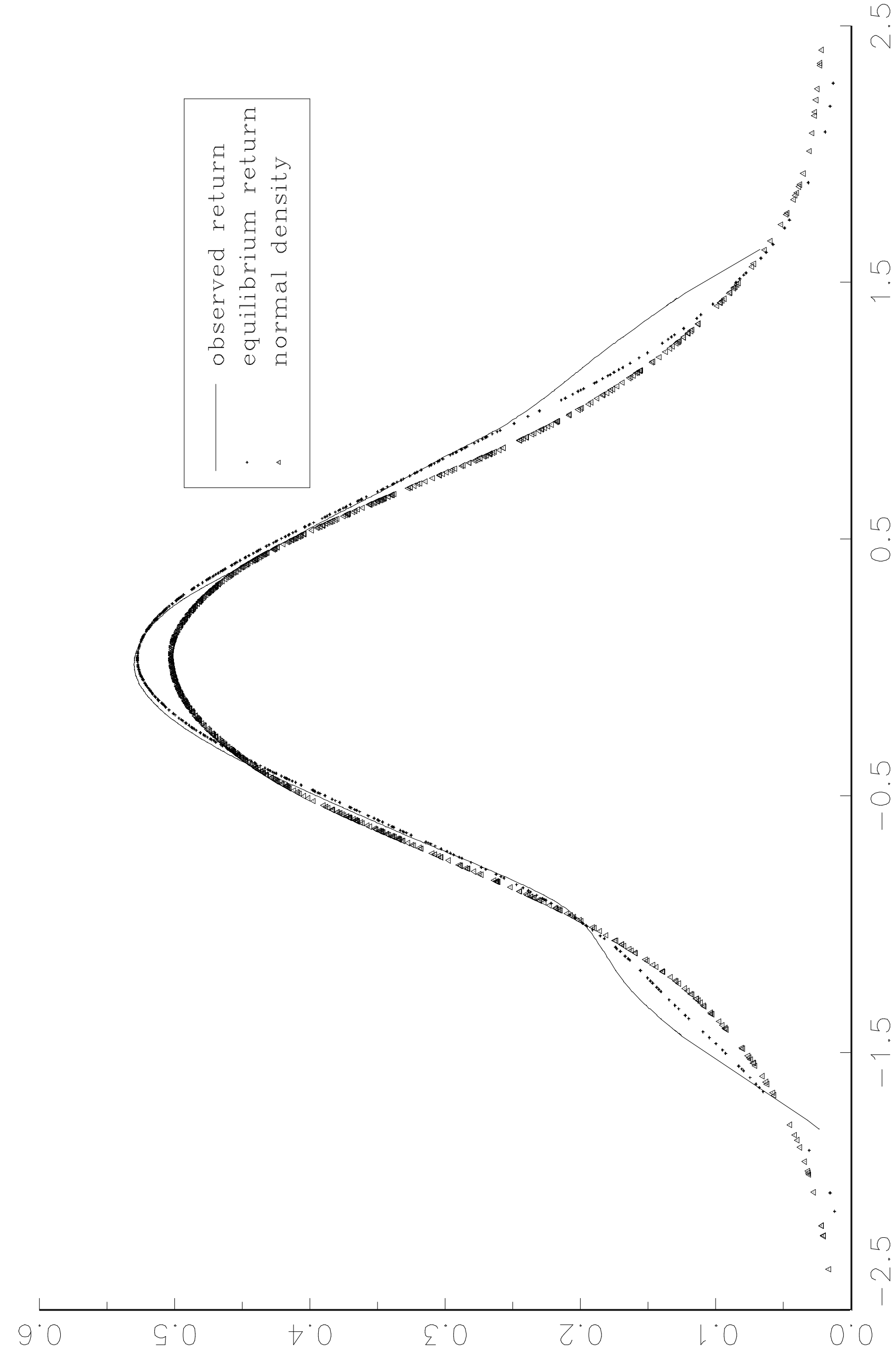


Figure 7: Histogram comparison of posterior parameters

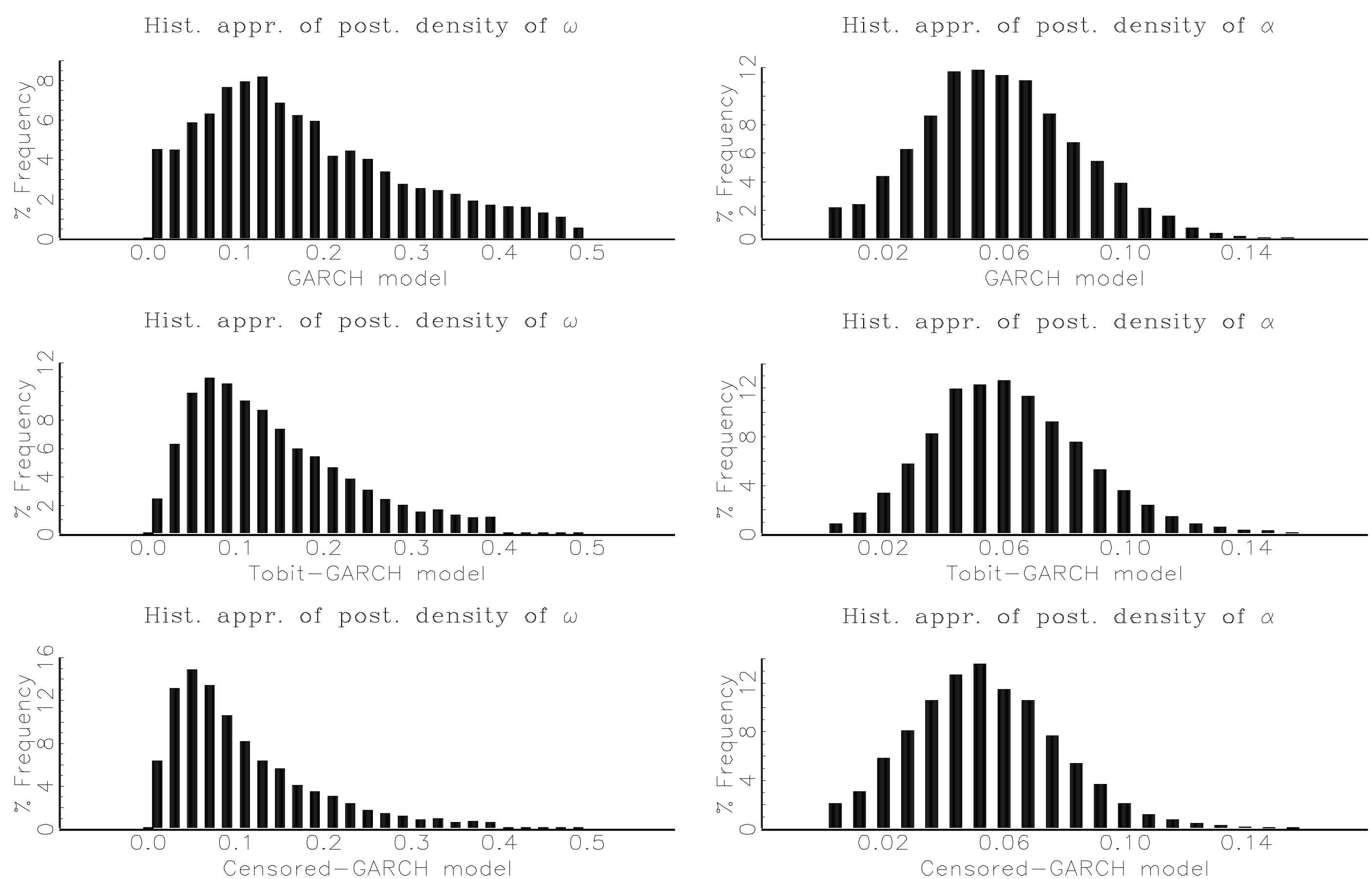
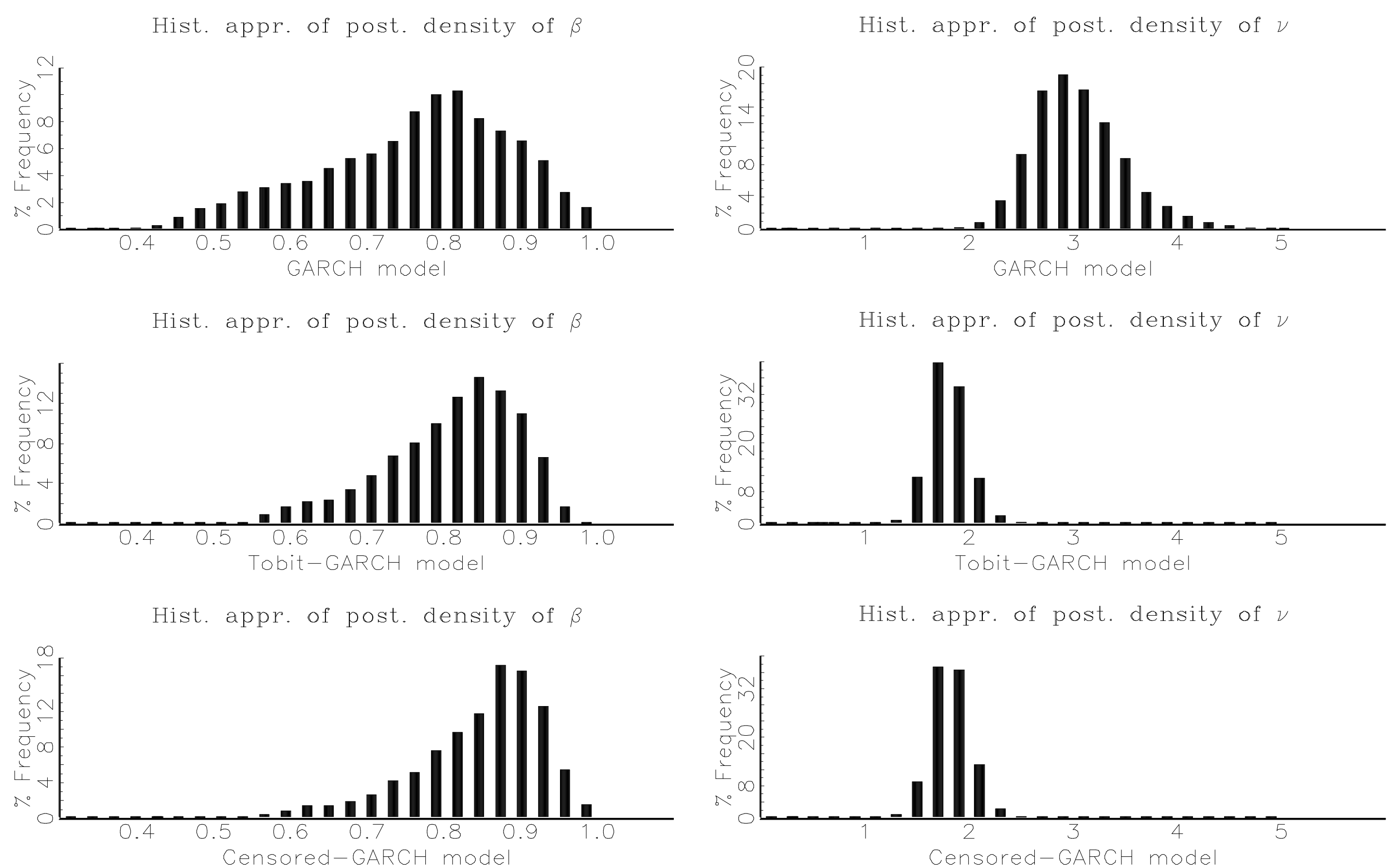


Figure 7: Histogram comparison of posterior parameters (continued)



Appendix

The algorithm of the griddy Gibbs sampler-data augmentation can be written as follows for M draws:

- *step 1*: initialize the chain at any value $\theta^{(0)}$ and $r^{*(0)}$ in the support of (θ, r^*) space.
- *step 2*: start the loop at $n = 1$.
- *step 3:* compute $p(\theta_1|\theta_2^{(n-1)}, r^{*(n-1)}, R)$ over the grid $(\zeta_1, \zeta_2, \cdots, \zeta_g)$ to obtain $G_p = (p_1, p_2, \cdots, p_g)$.
- *step 4*: compute the values $G_p = (0, \Phi_2, \cdots, \Phi_g)$ where

$$\Phi_i = \int_{\zeta_1}^{\zeta_i} p(\theta_1|\theta_2, r^*, R)\, d\theta_1 \quad i = 2, 3, \cdots, g,$$

  and normalize the G_p to get the cdf values G_p/Φ_g of $p(\theta_1|\theta_2^{(n-1)}, r^{*(n-1)}, R)$.
- *step 5*: generate $u \sim U(0,1)$ and invert the cdf G_p/Φ_g to get a draw $\theta_1^{(n)}$.
- *step 6*: redo step *3-6* for θ_2 and each element of latent data r^*.
- *step 7*: increment n by 1 and go to step *3* unless $n > M$.
- *step 8*: discard the initial m draws, and return all other draws.

In this algorithm, $p(\cdot)$ stands for the density function of the corresponding parameter or one element of the latent data conditioning on all other information.

A few comments are worth mentioning for the implementation of the algorithm. First, the choice of the grid of points is somewhat difficult and constitutes the main effort in applying the method. A proper choice of the grid points often requires the exploration of the shape of the conditional densities and a trial period. Second, the integration taken in step 4 can proceed in various ways. For simplicity, this paper uses the Simpson rule. In step 5, the inverse of the cdf is constructed by using linear interpolation.

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