

THE LANDAU POLE AT FINITE TEMPERATURE

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We study the Landau pole in the $\lambda\phi^4$ field theory at non-zero and large temperatures. We show that the position of the thermal Landau pole $\Lambda_L(T)$ is shifted to higher energies with respect to the zero temperature Landau pole $\Lambda_L(0)$. We find for high temperatures $T > \Lambda_L(0)$, $\Lambda_L(T) \simeq \pi^2 T / \log(T/\Lambda_L(0))$. Therefore, the range of applicability in energy of the $\lambda\phi^4$ field theory increases with the temperature.

As is well known the $\lambda\phi^4$ field theory as well as QED suffer from unphysical singularities at the Landau pole [1]. At zero temperature and resumming one-loop bubbles, this pole is found at an energy $\Lambda_L(0) = \mu e^{16\pi^2/[3\lambda]}$, where μ is the renormalization scale. In spite of that, $\lambda\phi^4$ can be used as an effective field theory for energy scales well below $\Lambda_L(0)$. In such a range, the theory makes full sense and contains a rich physics [2]. The Landau pole for very small coupling is at extremely high energies (beyond the Planck scale in the case of QED [1]) which reduces its relevance. In the context of the standard model the zero temperature Landau pole provides a limitation for the phenomenological validity of the theory through the triviality bound [3]. In addition, in the $O(4)$ linear sigma model where $\lambda/(4\pi) \sim 1$ the Landau pole again limits the domain of applicability of such effective theory.

It is therefore an important question to assess how the Landau pole moves when the temperature is non-zero, the Landau pole behaviour for high temperatures $T \gg \mu$ being especially relevant. To the best of our knowledge this problem has not been studied in the literature. We obtain in this note the Landau pole position as a function of the temperature.

The Landau pole and the phenomena linked to it clearly correspond to a scale of energy much higher than the renormalized mass for small coupling. Therefore, we can work in the massless Φ^4 theory without losing generality. Let us consider the critical theory with euclidean bare lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_0)^2 + \frac{1}{2}m_0^2 \Phi_0^2(x) + \frac{\lambda_0}{4!} \Phi_0^4(x), \quad (1)$$

where m_0^2 and λ_0 stand for the bare mass and coupling constant, respectively.

By massless theory, we mean a theory where the renormalized mass, defined as the zero momentum two-point function, exactly vanishes

$$m_{ren}^2(T) \equiv \Gamma_2(p^0 = 0, \vec{p} = 0; T) = 0, \quad (2)$$

at a given temperature T .

Our analysis on the running of the coupling constant $\lambda(q; T)$ remain valid in the massive theory provided that the renormalized thermal mass $m_{ren}(T) \neq 0$ is much smaller than T or that we consider momenta $q \gg m_{ren}(T)$.

The renormalized mass is derived from the self-consistent gap equation which has the form [4],

$$m_{ren}^2(T) = m_0^2 + \frac{1}{2}\lambda_0 T \sum_{n=-\infty}^{+\infty} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\nu_n^2 + q^2 + m_{ren}^2(T)}, \quad \nu_n = 2\pi nT. \quad (3)$$

We choose a negative bare mass $m_0^2 = -m^2 < 0$ in order to find a zero physical mass at some positive temperature. Setting $m_{ren}^2(T) = 0$ in eq.(3) yields,

$$0 = -m^2 + \frac{\lambda_0 T^2}{24}$$

Therefore, we choose from now on $m = \sqrt{\frac{\lambda_0}{24}} T$ in order to have a vanishing physical mass.

We want to study the evolution of the running coupling constant, defined from the four point vertex at the four-particle static symmetric point $\bar{p}_i = (0, \vec{p}_i)$. That is,

$$\bar{p}_i^2 = q^2, \quad \bar{p}_i \cdot \bar{p}_j = -q^2/3, \quad i \neq j, \quad i, j = 1, 2, 3, 4, \quad (4)$$

where $q > 0$ is a normalization mass scale.

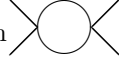
We have,

$$\lambda(q; T) = \Gamma_4(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; T) . \quad (5)$$

In particular we define the zero-temperature μ -momentum renormalized coupling constant as

$$\lambda(\mu; 0) = \lambda = \Gamma_4(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; 0) |_{q=\mu} . \quad (6)$$

We notice that there is a certain freedom in the choice of the thermal coupling constant. Other prescriptions besides eq.(5) are possible as well.

At the one-loop level the four point function is given by the bubble diagram  plus the two crossed bubble graphs; we have the explicit expression at the symmetric point,

$$I_{bub}(\nu_n, q; T) = \frac{3}{2} T \sum_{m=-\infty}^{+\infty} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\nu_m^2 + \vec{p}^2} \frac{1}{(\nu_m + \nu_n)^2 + (\vec{p} + \vec{q})^2} , \quad (7)$$

where a $\frac{1}{2}$ comes from the symmetry factor of the bubble diagrams.

We renormalize I_{bub} with a zero-temperature μ -momentum subtraction as

$$\Pi(\nu_n, q) = I_{bub}(\nu_n, q; T) - I_{bub}(0, \mu; 0) . \quad (8)$$

For critical theories the renormalized bubble diagram $\Pi(s, q; T)$ can be analytically computed for any temperature, within the imaginary time formalism with analytic continuation to complex $s = \nu_n$, or within the real time formalism [5] with the result

$$\Pi(s, q; T) = \frac{3}{(4\pi)^2} \left\{ \ln \left(\frac{\mu}{4\pi e T} \right) + \frac{s}{q} \operatorname{arctg} \frac{q}{s} - \frac{i\pi T}{q} \log \left[\frac{\Gamma \left(\frac{is+q}{4\pi iT} \right) \Gamma \left(1 + \frac{is+q}{4\pi iT} \right)}{\Gamma \left(\frac{is-q}{4\pi iT} \right) \Gamma \left(1 + \frac{is-q}{4\pi iT} \right)} \right] \right\} , \quad (9)$$

where $e = 2.7182818285 \dots$ is the basis of natural logarithms.

For non-critical theories there are corrections to this result of the order $m_{ren}(T)/T$. The four point vertex can be perturbatively evaluated as

$$\Gamma_4(\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4; T) = \lambda - \lambda^2 \Pi(0, q) + O(\lambda^3) . \quad (10)$$

The one-loop resummed thermal running coupling constant $\lambda(q; T)$ as obtained from renormalization group analysis takes the form

$$\lambda(q; T) = \frac{\lambda}{1 + \lambda \Pi(0, q; T)} . \quad (11)$$

The explicit form is

$$\lambda(q; T) = \frac{\lambda}{1 + \frac{3\lambda}{(4\pi)^2} \left\{ \ln \frac{\mu}{4\pi e T} + \frac{\pi T}{q} \left[-\pi + 4 \operatorname{Im} \ln \Gamma \left(\frac{q}{4\pi iT} \right) \right] \right\}} . \quad (12)$$

Notice that $\lambda(q; T)$ is a function of q/T (and μ/T). The low and high temperature limits follow using the Stirling formula and small argument expansions for the Gamma function [6]

$$\text{Im} \ln \Gamma(-ix) \stackrel{x \rightarrow 0^+}{=} \frac{\pi}{2} + \gamma x + \mathcal{O}(x^2) \quad , \quad (13)$$

and

$$\text{Im} \ln \Gamma(-ix) \stackrel{x \rightarrow \pm\infty}{=} -x \ln \frac{|x|}{e} + \frac{\pi}{4} + \frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right) \quad , \quad (14)$$

where $\gamma = 0.5772157 \dots$ stands for the Euler-Mascheroni constant.

We find for soft momenta $q \ll T$,

$$\lambda(q, T) = \frac{16}{3} \frac{q}{T} + \mathcal{O}\left(\frac{q^2}{T^2} \log \frac{T}{\mu}\right) \quad (15)$$

i.e. the interaction linearly vanishes at low momenta due to the dimensional reduction to a three-dimensional theory, as expected.

At zero-temperature we recover from eq.(12) the usual zero-temperature resummed coupling constant [2]

$$\lambda(q; 0) = \frac{\lambda}{1 + \frac{3}{32\pi^2} \lambda \ln \frac{\mu^2}{q^2}} \quad (16)$$

which exhibits the Landau pole at the scale

$$\Lambda_L(0) \equiv \Lambda_0 = \mu e^{16\pi^2/[3\lambda]} \quad . \quad (17)$$

In the context of effective field theories, this scale is interpreted as the scale where new physics enters, i.e. the scale of the order of the ultraviolet cutoff of the theory Λ where irrelevant operators play an important role and therefore the theory defined by the lagrangian (1) breaks down.

A renormalization group resummed perturbation theory can be consistently implemented only when the internal momenta are integrate up to a scale of order $\Lambda \leq \Lambda_0$.

We would like to know how thermal effects affect the Landau pole. In particular where is the position of the thermal Landau pole $\Lambda_L(T)$ at temperature T . This is the principal outcome of our analysis.

We define the thermal Landau point $\Lambda_L(T)$ as the zero of the denominator in eq.(12) for the resummed coupling $\lambda(q; T)$. That is, $\Lambda_L(T)$ is the solution of the equation,

$$\frac{\pi - 4 \text{Im} \ln \Gamma\left(\frac{\Lambda_L(T)}{4\pi i T}\right)}{\Lambda_L(T)} = \frac{1}{\pi T} \left[\frac{(4\pi)^2}{3\lambda} + \ln\left(\frac{\mu}{4\pi e T}\right) \right] \quad . \quad (18)$$

This is a transcendental equation which in general can be only solved numerically. However, in the high ($T \geq \Lambda_0$) and low ($T \ll \Lambda_0$) temperature limits, the position of the Landau pole can be determined analytically using eqs.(13) and (14), respectively.

We find for temperatures below the zero temperature Landau pole $T \ll \Lambda_0$

$$\Lambda_L(T) \stackrel{T \ll \Lambda_0}{=} \Lambda_0 + \frac{4\pi^2}{3\Lambda_0} T^2 + \mathcal{O}\left(\frac{T^3}{\Lambda_0^2}\right) \quad . \quad (19)$$

That is, the temperature **increases** as T^2/Λ_0 the position of the Landau pole in energy above its zero temperature value.

We find for temperatures above the zero temperature Landau pole $T \gtrsim \Lambda_0$,

$$\Lambda_L(T) \stackrel{T \gtrsim \Lambda_0}{=} \frac{\pi^2 T}{\ln\left(\frac{4\pi e T}{\Lambda_0}\right) - \gamma + \mathcal{O}\left(\frac{\Lambda_0}{T}\right)} \quad . \quad (20)$$

We see that the thermal Landau pole position grows linearly as $T/\ln T$ and it is therefore **well above** the zero temperature Landau pole.

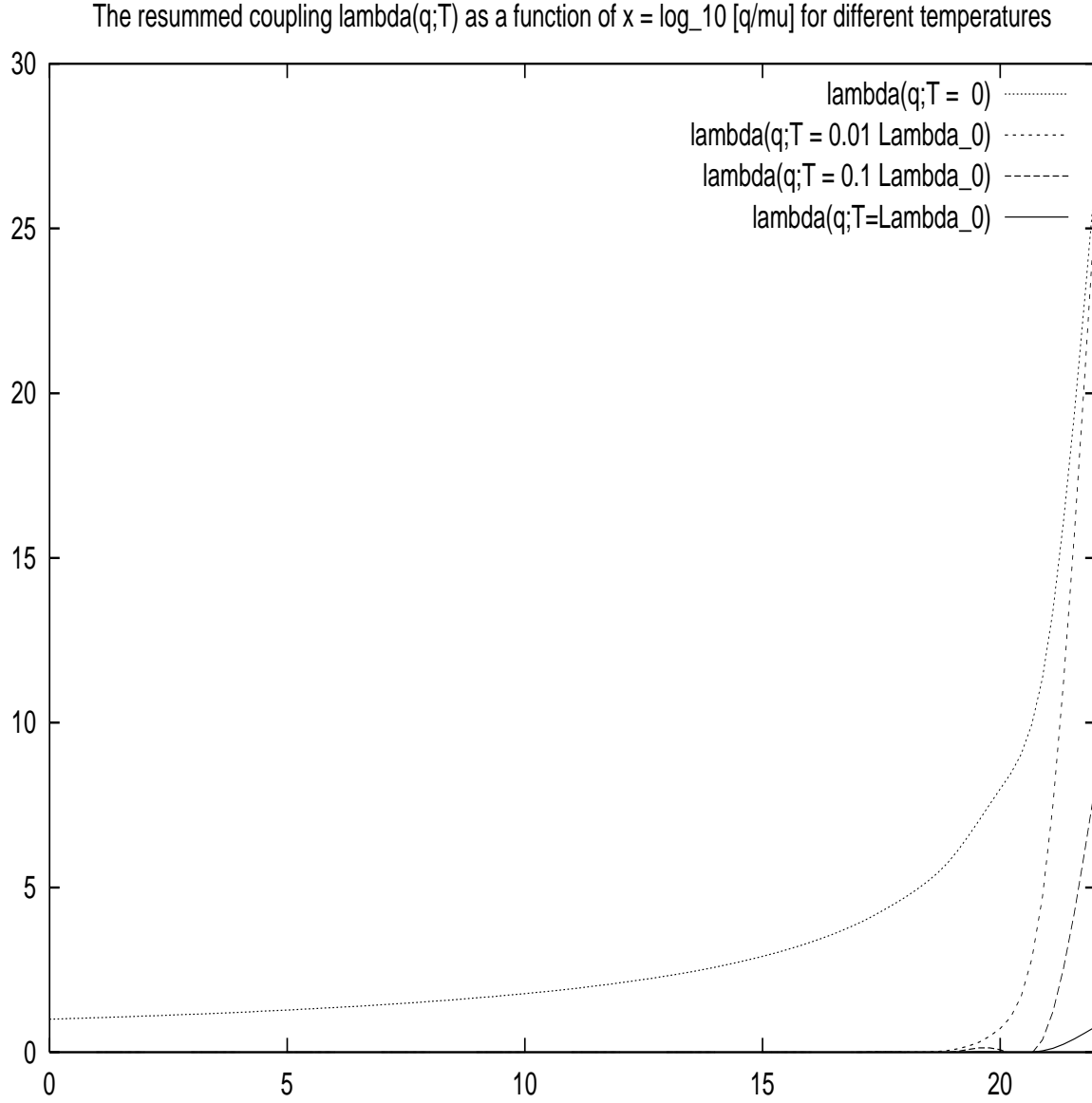


FIG. 1. The resummed coupling constant $\lambda(q;T)$ as a function of $x \equiv \log_{10} \frac{q}{\mu}$ for four different temperatures $T = 0$ (upmost line), $T = 0.01 \Lambda_0$ (second line from above), $T = 0.1 \Lambda_0$ (third line from above) and $T = \Lambda_0$ (bottom line). The curves correspond to the renormalization prescription $\lambda(\mu;0) = \lambda = 1$, such that the zero temperature Landau pole is at the scale $\Lambda_0 = 7.25 \cdot 10^{22} \mu$. At high temperatures the increasing of the coupling constant at large momenta is less pronounced than at zero-temperature and the Landau pole is shifted to higher momentum scales.

We notice that the analytic formulas (19) and (20) are in excellent agreement with the numerical evaluation using eq.(12) even beyond their expected range of validity. In particular, eq.(20) reproduces eq.(18) with better than 1% accuracy at $T = \Lambda_0$ and $\lambda = 1$.

In figure 1 we display the evolution of the resummed coupling $\lambda(q; T)$ given by eq.(12) for different temperatures starting from $q = \mu$ up to $q = 10^{22}\mu$ for $\lambda = 1$. In figure 2 we display the position of the thermal Landau pole for different temperatures and $\lambda = 1$.

In summary, both in low T and in the high T region the thermal effect consists in pushing the Landau pole scale to higher energies than the zero temperature Landau pole. This is true for all temperatures.

The Landau pole behaves for high and low temperatures in very different ways due to the different runnings of the coupling. For high temperature dimensional reduction takes place yielding a dimensionful three dimensional coupling λT . Therefore, the one-loop contribution to the effective coupling at momentum q behaves as T/q . This follows just by dimensional considerations or by inspecting the one-loop diagram with two propagators integrated over three-dimensional loop momenta. The Landau pole appears when the one-loop contribution balances the tree level contribution and now this can happen when $q \simeq T$. This gives a simple understanding why the Landau pole behaves as in eq.(20) for high temperature.

Since the Landau pole grows in energy when the temperature grows, the range of validity of Φ^4 as an effective theory does increase. As mentioned at the begining, the Landau pole sets the limit of validity for theories as the standard model [3] and the $O(4)$ linear sigma model.

As is known, fermionic systems often exhibit behaviours which differ from those of bosonic systems (see for example [7]). We find in spinor electrodynamics that that the Landau pole is well above T for temperatures above the Landau pole at zero temperature. In addition, the Landau pole position grows with T in such regime [work in progress]. Therefore, the Landau pole behaviour in QED seems analogous to the scalar case.

Finally we notice that the conclusions of this paper trivially generalize to $O(N)$ -models with interaction $\frac{\lambda'}{2N}[\vec{\Phi}^2(x)]^2$ through a trivial rescaling of the coupling constant by a factor three. It is interesting to consider the large N limit since in this limit the present results on the Landau pole are *exact*, even in the strong coupling regime.

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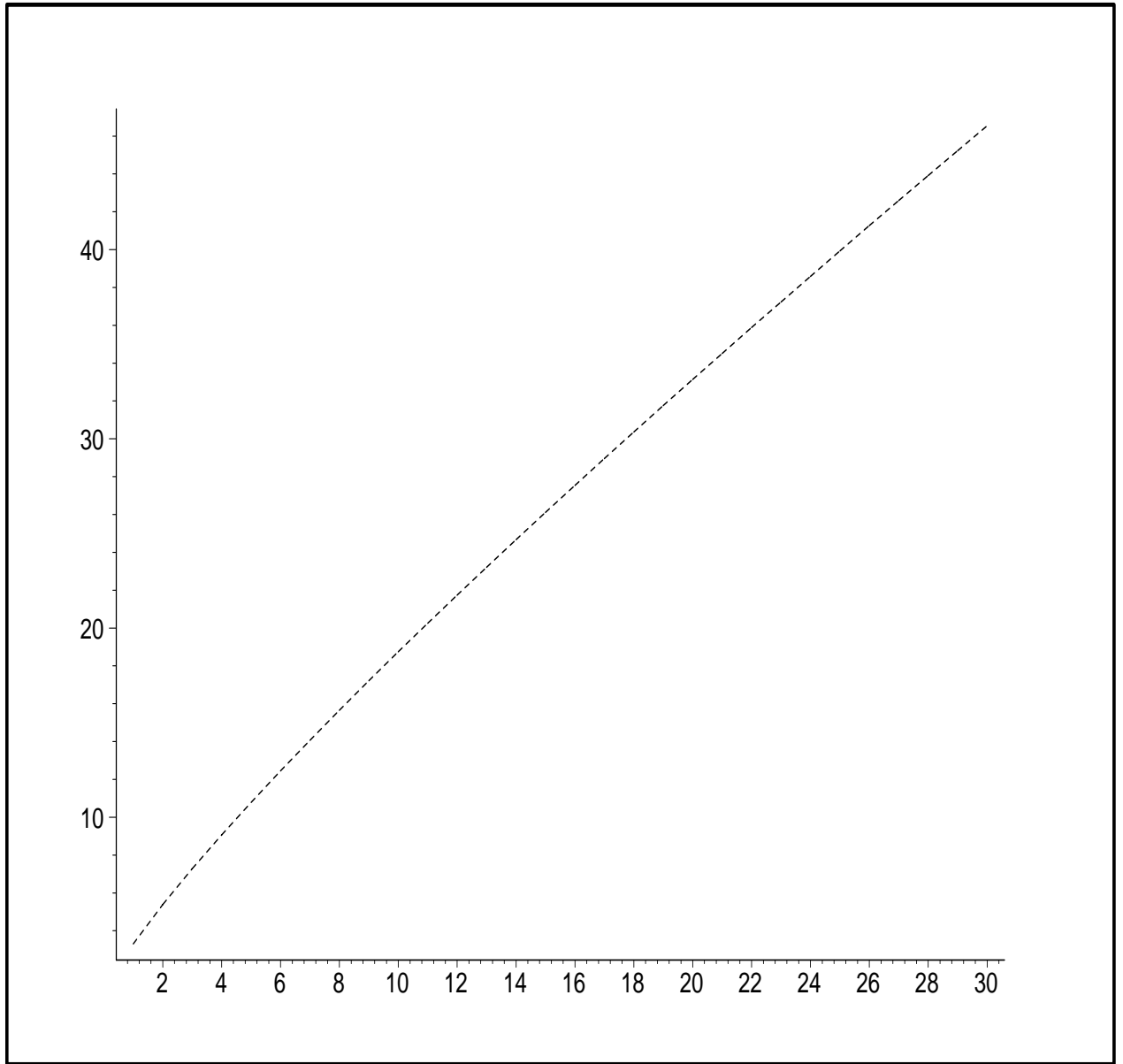


FIG. 2. Position of the Landau pole $\Lambda_L(T)$ in units of the zero temperature pole Λ_0 as a function of temperature from $T = \Lambda_0$ to $T = 30\Lambda_0$. Here $\lambda(\mu; 0) = \lambda = 1$.