CHANGE OF VARIABLE IN DOUBLE INTEGRALS
-THE JACOBIAN MATRIX-

→ THEOREM

LET R $\wedge$ S BE REGIONS IN THE uv $\wedge$ xy PLANES RESPECTIVALLY. SUPPOSE

$$\vec{T}: R \rightarrow S, \quad \vec{T}(u,v) = \langle x(u,v), y(u,v) \rangle$$

IS A COORDINATE TRANSFORM OF CLASS C' THAT MAPS R ONTO S IN A ONE-ONE FASHION. IF $f: S \rightarrow \mathbb{R}$ IS AN INTEGRABE FUNCTION, THEN

$$\iint_S f(x,y)\,dA = \iint_R f[x(u,v), y(u,v)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

WHERE THE JACOBIAN

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \equiv \| \vec{T}_u(u,v) \times \vec{T}_v(u,v) \|$$

- PRELIMINARY NOTATION:

1) $\Delta S_{ij} \equiv \text{AREA}(S_{ij})$

2) $$\sum(1) \equiv \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta S_{ij}$$

3) $$\sum(2) \equiv \sum_{i=1}^{m} \sum_{j=1}^{n} f[x(u,v), y(u,v)] \| \vec{r}_u(u_{i-1}, v_{j-1}) \times \vec{r}_v(u_{i-1}, v_{j-1}) \| \Delta v_j \Delta u_i$$

4) $$\lim \Rightarrow \lim_{n,m \to \infty} \equiv \lim_{|P| \to 0}$$

5) $\vec{a}_{ij} \equiv \vec{r}(u_{i-1} + \Delta u_i, v_{j-1}) - \vec{r}(u_{i-1}, v_{j-1})$

6) $\vec{b}_{ij} \equiv \vec{r}(u_{i-1}, v_{j-1} + \Delta v_j) - \vec{r}(u_{i-1}, v_{j-1})$

- PROOF

— WE BEGIN BY PROVING THE THEOREM IN THE SPECIAL CASE WHEREIN S IS THE IMAGE OF A RECTANGULAR REGION R THRU THE C^1, 1-1 TRANSFORM, $\vec{T}: R \to S$.

$$R = \{(u, v) \mid \alpha \leq u \leq \beta \wedge \gamma \leq v \leq \varsigma\}$$

— WE CONSIDER "REGULAR" PARTITIONS WHEREIN $[\alpha, \beta]$ IS DIVIDED INTO m INTERVALS OF EQUAL WIDTH $\wedge$ $[\gamma, \varsigma]$ IS DIVIDED INTO n INTERVALS OF EQUAL WIDTH.

FOR FIXED $m, n \in \mathbb{N}$, $\exists$ mn SUBREGIONS WITHIN R WE MAY CALL R_{ij}, $\wedge$ FOR EACH R_{ij} $\exists$ A COOROSPONDING IMAGE IN S WE MAY CALL S_{ij} !

$$S_{ij} = \vec{T}(R_{ij})$$

— WE PRESCRIBE A RULE OF ASSIGNMENT. FOR EACH R_{ij}, WE CHOOSE THE LOWER LEFT CORNER !

$$P_{ij} \equiv (u_{i-1}, v_{j-1})$$

NOTE THAT THE POINT P_{ij} COOROSPONDS TO A POINT IN THE xy-PLANE LAYING AT SOME CORNER OF S_{ij} !

$$\vec{T}(u_{i-1}, v_{j-1}) \Rightarrow (x_{ij}, y_{ij}) \equiv P'_{ij}$$

— WITH THE NOTATION', PARTITIONING $\wedge$ R.O.A. ARGUMENTS MADE, WE MAY SUBMIT PROOF VIA THE FOLLOWING TWO LEMMAS !

→ LEMMA #1

$\lim \sum(1) = \lim \sum(2)$

- PROOF OF LEMMA #1

— LEMMA #1 MAY BE PROVEN ANALYTICALLY BY SHOWING THE FOLLOWING STATEMENT HOLDS

(ρ) $\forall \epsilon > 0, \exists \lambda \in \mathbb{N} \mid$

$$\left| \sum(1) - \sum(2) \right| < \epsilon$$

if $m, n > \lambda$

— BECAUSE $\vec{T}$ IS OF CLASS C^1, AS $|P| \to 0$ OVER THE REGION R, EACH $S_{ij} \cong \square$, AS EACH OF THE FOUR COORDINATE CURVES BOUNDING S_{ij} ARE CERTAINLY $\approx$ CONGRUENT TO LINES OVER $\Delta v_i \wedge \Delta v_j$. WE MAY WRITE

(R_1) $\Delta S_{ij} = \text{AREA}(\square_{ij}) + \epsilon_1(|P|)$

— A $\square_{ij}$ SPANNING S_{ij} MAY BE FOUND EASILY, FOR IT IS SPANNED BY $\vec{a}_{ij} \wedge \vec{b}_{ij}$. $\therefore$ $\text{AREA}(\square_{ij}) = \| \vec{a}_{ij} \times \vec{b}_{ij} \|$. NOTE THAT

$\vec{a}_{ij} \approx \Delta v_i \vec{T}_v(P_{ij}) \wedge \vec{b}_{ij} \approx \Delta v_j \vec{T}_v(P_{ij})$

SINCE AS $|P| \to 0$, THE ABOVE $\approx \to =$, WE MAY WRITE

(R_2) $\text{AREA}(\square_{ij}) = \| \vec{T}_v(v_{i-1}, v_{j-1}) \times \vec{T}_v(v_{i-1}, v_{j-1}) \| \Delta v_j \Delta v_i + \epsilon_2(|P|)$

- SINCE $|P| \to 0 \Rightarrow \epsilon_{1,2} \to 0$, IT IS CLEAR THAT FOR ANY $\epsilon > 0$, $\exists \lambda \in \mathbb{N}$ | IF $m, n > \lambda$, THE INEQUALITY IN (ρ) HOLDS TRUE. THIS IMPLIES THAT

$$\lim \sum (1) = \lim \sum (2)$$

AS DESIRED.

Q.E.D.

→ LEMMA #2

$$\iint_S f(x,y)\,dA = \lim \sum (1)$$

- PROOF OF LEMMA #2

- LEMMA #2 MAY BE PROVEN ANALYTICALLY BY SHOWING THE FOLLOWING STATEMENT HOLDS

(ρ) $\forall \epsilon > 0$, $\exists \lambda \in \mathbb{N}$ |

$$\left| \iint_S f(x,y)\,dA - \sum (1) \right| < \epsilon$$

if $m, n > \lambda$

- OBVIOUSLY WE MAY MAKE $\sum(1)$ ARBITRARALLY CLOSE TO $\iint_S f(x,y)\,dA$ BY CHOOSING P TO BE SUFFICENTALLY FINE OVER $\mathcal{R}$. So (ρ) HOLDS.

Q.E.D.

— WE CONCLUDE, BY LEMMAS 1 ∧ 2,

$$(R_3) \quad \iint_S f(x,y)\,dA = \lim \sum (2)$$

$$= \iint_R f[x(u,v), y(u,v)] \| \vec{T}_u(u,v) \times \vec{T}_v(u,v) \| \, dA_{uv}$$

$$= \iint_R f[x(u,v), y(u,v)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA_{uv}$$

— (R_3) MAY BE GENERALIZED USING FUBINI'S METHODOLOGY.

Q.E.D.