CHAPTER 3

FINITE DIFFERENCE APPROXIMATION TO THE MULTIGROUP DIFFUSION IN FINITE CYLINDRICAL REACTOR

The steady state neutron diffusion equation is written as

$$D_g(\mathrm{r,z})\nabla_g^2\varphi_g(\mathrm{r,z}) - \textstyle\sum_{ag}(r,z)\varphi_g(\mathrm{r,z}) - \displaystyle\sum_{h=g+1}^{N} \textstyle\sum_{g\to h}(r,z)\,\varphi_g(\mathrm{r,z}) + \displaystyle\sum_{h=1}^{g-1} \textstyle\sum_{h\to g}(r,z)\varphi_g(\mathrm{r,z}) + \chi_g \displaystyle\sum_{h=1}^{N} \nu_g \Sigma_{fh}(r,z)\,\varphi_h(\mathrm{r,z}) = 0 \qquad 3.1$$

$$-D_g(r,z)\nabla^2\varphi_g(r,z) + \textstyle\sum_{trg}(r,z)\varphi_g(r,z) = S_g \qquad 3.2$$

Where

$$\textstyle\sum_{trg}(r,z)\varphi_g(r,z) = \textstyle\sum_{ag}(r,z)\varphi_g(\mathrm{r,z}) - \displaystyle\sum_{h=g+1}^{N} \textstyle\sum_{g\to h}(r,z)\,\varphi_g(\mathrm{r,z}) + \displaystyle\sum_{h=1}^{g-1} \textstyle\sum_{h\to g}(r,z)\varphi_g(\mathrm{r,z}) \qquad 3.3$$

Also,

$$S_g = \frac{\chi_g}{K_{eff}} \sum_{h=1}^{N} \nu_g \Sigma_{fh}(r,z)\,\varphi_h(\mathrm{r,z}) \qquad 3.4$$

$D_g(\mathrm{r,z})$ = The group diffusion coefficient

∇^2 = The Laplace operator of the group

$\varphi_g(\mathrm{r,z})$ = The group flux
$\sum_{ag}(r,z)$ = total macroscopic absorption cross-section for the group
$\sum_{g\to h}(r,z)$ =total macroscopic scattering cross-section from one group g to h
$\sum_{h\to g}(r,z)$ = total macroscopic scattering cross-section from group h to g

$\sum_{trg}(r,z)$=$\sum_{tg}(r,z)$ = total group macroscopic removal cross-section
χ_g = the fraction of fission neutrons that are emitted with energies in the gth group
ν_g = the average number of fission neutrons released as the result of fissions induced by neutrons in the gth group
$\sum_{fh}(r,z)$ = the group-averaged macroscopic fission cross-section

In a finite cylinder, the equation becomes and dropping the subscript g

$$\frac{1}{r}\frac{\partial}{\partial r}\left[D(r,z)r\frac{\partial\varphi(r,z)}{\partial r}\right]+\frac{\partial}{\partial z}\left[D(r,z)\frac{\partial\varphi(r,z)}{\partial z}\right]-\sum_t(r,z)\varphi(r,z)=-S(r,z) \tag{3.5}$$

Multiplying through by r

$$\frac{\partial}{\partial r}\left[D(r,z)r\frac{\partial\varphi(r,z)}{\partial r}\right]+\frac{\partial}{\partial z}\left[rD(r,z)\frac{\partial\varphi(r,z)}{\partial z}\right]-r\sum_t(r,z)\varphi(r,z)=-rS(r,z) \tag{3.6}$$

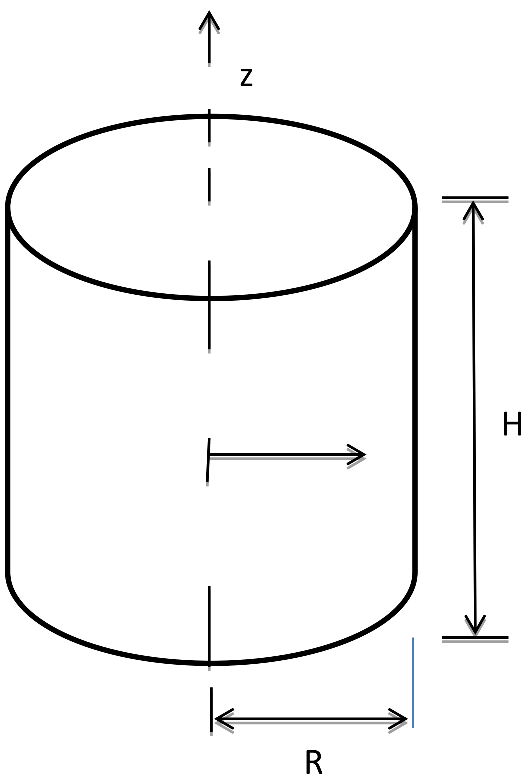
We can define J and Y as

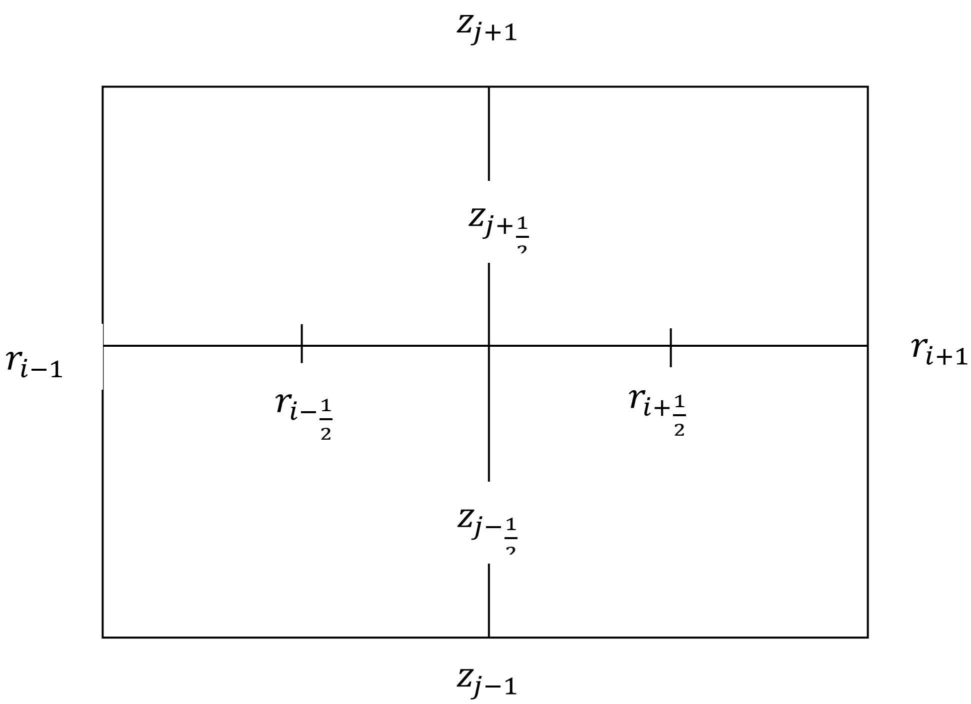
$$J=D(r,z)\frac{\partial\varphi(r,z)}{\partial r} \tag{3.7}$$

$$Y=D(r,z)\frac{\partial\varphi(r,z)}{\partial z} \tag{3.8}$$

$$r\frac{\partial J}{\partial r}+r\frac{\partial y}{\partial z}-r\sum_t(r,z)\varphi(r,z)=-rS(r,z) \tag{3.9}$$

The reactor is assumed to cover a special mesh in the r and z dimensions as shown in the figure below





$$r\int_{r_{i-1/2}}^{r_{i+1/2}}\int_{z_{j-1/2}}^{z_{i+1/2}}\frac{\partial J}{\partial r}drdz+\int_{r_{i-1/2}}^{r_{i+1/2}}\int_{z_{j-1/2}}^{z_{i+1/2}}\frac{\partial Y}{\partial r}drdz$$

$$-r\iint\int_{r_{i-1/2}}^{r_{i+1/2}}\int_{z_{j-1/2}}^{z_{i+1/2}}\Sigma_t\varphi(r,z)drdz$$

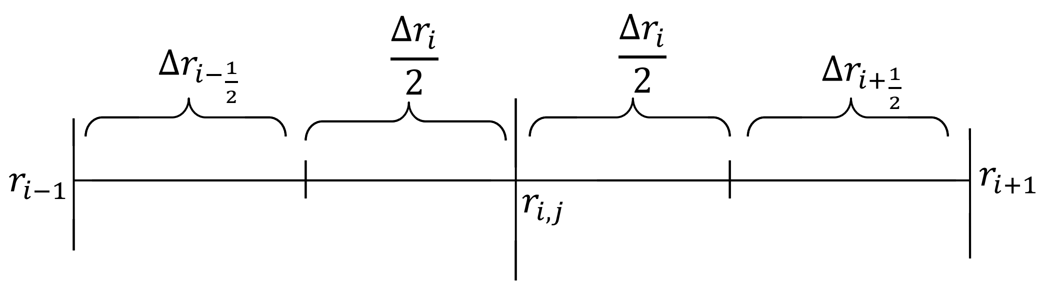
$$=-r\int_{r_{i-1/2}}^{r_{i+1/2}}\int_{z_{j-1/2}}^{z_{i+1/2}}S(r,z)drdz \qquad 3.10$$

$$r_i[J_{i+1/2}-J_{i-1/2}]\Delta z+[Y_{i+1/2}-Y_{i-1/2}]r_i\Delta r-r_{i,j}\ \sum\nolimits_{t_{i,j}}\varphi_{i,j}\Delta r\Delta z$$

$$=-r_iS_{i,j}\Delta r\Delta z$$

3.11

Neutron current balance in r-axis



$$J_{\frac{i+1}{2}} \;=\; J_{\frac{i+1}{2}}+ \;=\; J_{\frac{i+1}{2}}- \qquad 3.12$$

$$J_{\frac{i+1}{2}}+ = D_{i+1,j}\frac{[\varphi_{i+1,j}-\varphi_{i,j}]}{\Delta r_{\frac{i+1}{2}}} \qquad 3.13$$

$$J_{\frac{i+1}{2}}- = D_{i,j}\frac{\left[\varphi_{\frac{i+1}{2},j}-\varphi_{i,j}\right]}{\frac{\Delta r_i}{2}} \qquad 3.14$$

$$2D_{i+1,j}\frac{[\varphi_{i+1,j}-\varphi_{\frac{i+1}{2},j}]}{\Delta r_{i+1}} = 2D_{i,j}\frac{[\varphi_{\frac{i+1}{2},j}-\varphi_{i,j}]}{\Delta r_i}$$

$$D_{i+1,j}\Delta r_i\varphi_{i+1,j} - D_{i+1,j}\Delta r_i\varphi_{\frac{i+1}{2},j} = D_{i,j}\Delta r_{i+1}\varphi_{\frac{i+1}{2},j} - D_{i,j}\Delta r_{i+1}\varphi_{i,j}$$

$$D_{i,j}\Delta r_{i+1}\varphi_{\ i+1/2,j} + D_{i+1,j}\Delta r_i\varphi_{i+1/2,j} = D_{i+1,j}\Delta r_i\varphi_{i+1,j} + D_{i,j}\Delta r_{i+1}\varphi_{i,j}$$

$$\varphi_{\frac{i+1}{2},j} = \frac{D_{i+1,j}\Delta r_i\varphi_{i+1,j} + D_{i,j}\Delta r_{i+1}\varphi_{i,j}}{D_{i,j}\Delta r_{i+1} + D_{i+1,j}\Delta r_i} \qquad 3.15$$

Put equation 3.15 in 3.13

$$J_{\frac{i+1}{2}} = \frac{2D_{i+1,j}}{\Delta r_{i+1}}\left[\frac{\varphi_{i+1,j}D_{i,j}\Delta r_{i+1} + \varphi_{i+1,j}D_{i+1}\Delta r_i - D_{i+1,j}\Delta r_i\varphi_{i+1,j} - D_{i,j}\Delta r_{i+1}\varphi_{i,j}}{D_{i,j}\Delta r_{i+1} + D_{i+1,j}\Delta r_i}\right]$$

$$J_{\frac{i+1}{2}} = \frac{2D_{i+1,j}}{\Delta r_{i+1}}\left[\frac{\varphi_{i+1,j}D_{i,j}\Delta r_{i+1} - D_{i,j}\Delta r_{i+1}\varphi_{i,j}}{D_{i,j}\Delta r_{i+1} + D_{i+1,j}\Delta r_i}\right]$$

$$J_{\frac{i+1}{2}} = 2D_{i+1,j}\left[\frac{D_{i,j}\varphi_{i+1,j} - D_{i,j}\varphi_{i,j}}{D_{i,j}\Delta r_{i+1} + D_{i+1,j}\Delta r_i}\right]$$

Then

$$J_{\frac{i+1}{2}} = \frac{2D_{i+1,j}D_{i,j}}{D_{i+1,j}\Delta r_i + D_{i,j}\Delta r_{i+1}}\left[\varphi_{i+1,j} - \varphi_{i,j}\right] \qquad 3.16$$

$$J_{\frac{i-1}{2}} \quad = \quad J_{\frac{i-1}{2}}+ \;=\; J_{\frac{i-1}{2}}- \qquad 3.17$$

$$J_{\frac{i-1}{2}}+ = D_{i,j}\left[\frac{\varphi_{i,j} - \varphi_{\frac{i-1}{2},j}}{\frac{\Delta r_i}{2}}\right] \qquad 3.18$$

$$J_{\frac{i-1}{2}}- \;= D_{i-1,j}\left[\frac{\varphi_{\frac{i-1}{2},j} - \varphi_{i-1,j}}{\Delta r_{\frac{i-1}{2}}}\right] \qquad 3.19$$

$$2D_{i-1,j}\frac{2D_{i-1,j}\left[\varphi_{\frac{i-1}{2},j} - \varphi_{i-1,j}\right]}{\Delta r_{i-1}} = 2D_{i,j}\frac{\varphi_{i,j} - \varphi_{\frac{i-1}{2},j}}{\frac{\Delta r_i}{2}}$$

$$D_{i-1,j}\Delta r_i\varphi_{\frac{i-1}{2},j} - D_{i-1,j}\Delta r_i\varphi_{i-1,j} = D_{i,j}\Delta r_{i-1}\varphi_{i,1,j} - D_{i,j}\Delta r_{i-1}\varphi_{\frac{i-1}{2},j}$$

$$D_{i-1,j}\Delta r_i\varphi_{\frac{i-1}{2},j} + D_{i,j}\Delta r_{i-1}\varphi_{\frac{i-1}{2},j} = D_{i,j}\Delta r_{i-1}\varphi_{i,j} + D_{i-1,j}\Delta r_{i+1}\varphi_{i-1,j}$$

$$\varphi_{\frac{i-1}{2},j} = \frac{D_{i,j}\Delta r_{i-1}\varphi_{i,j} + D_{i-1,j}\Delta r_i\varphi_{i-1,j}}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}} \qquad 3.20$$

Put 3.20 in 3.18

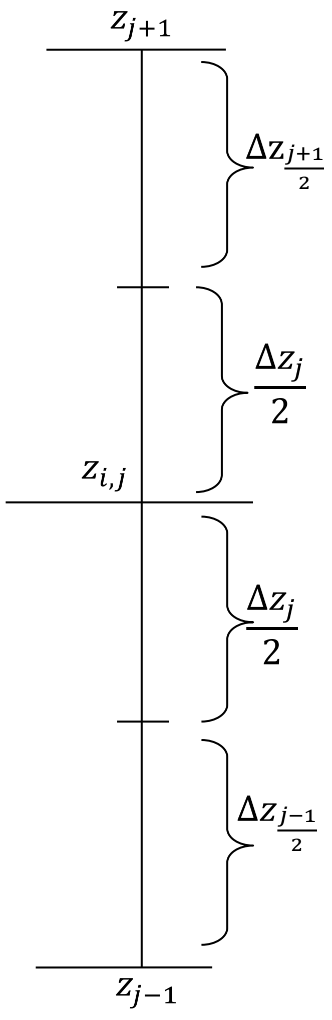
$$J_{\frac{i-1}{2}}$$

$$= \frac{2D_{i-1,j}}{\Delta r_{i-1}}\left[\frac{D_{i,j}\Delta r_{i-1}\varphi_{i,j} + D_{i-1,j}\Delta r_i\varphi_{i-1,j} - D_{i-1,j}\Delta r_i\varphi_{i-1,j} - D_{i,j}\Delta r_{i-1}\varphi_{i-1,j}}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}}\right]$$

$$J_{\frac{i-1}{2}} = \frac{2D_{i-1,j}}{\Delta r_{i-1}}\left[\frac{D_{i,j}\Delta r_{i-1}\varphi_{i,j} - D_{i,j}\Delta r_{i-1}\varphi_{i-1,j}}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}}\right]$$

$$J_{\frac{i-1}{2}} = \frac{2D_{i-1,j}D_{i,j}}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}}\left(\varphi_{i,j} - \varphi_{i-1,j}\right) \qquad 3.21$$

Current balance in the +z direction



$$\gamma_{\frac{j+1}{2}} = \gamma_{\frac{j+1}{2}}+ \; = \; \gamma_{\frac{j+1}{2}} - \qquad 3.22$$

$$\gamma_{\frac{j+1}{2}}+ \; = \frac{D_{i,j+1}[\varphi_{i,j+1} - \varphi_{i,\frac{j+1}{2}}]}{\Delta z_{\frac{j+1}{2}}} \qquad 3.23$$

$$\gamma_{\frac{j+1}{2}} - = \frac{D_{i,j}\left[\varphi_{i,j+1} - \varphi_{i,j}\right]}{\frac{\Delta z_j}{2}} \qquad 3.24$$

$$\frac{2D_{i,j+1}\left[\varphi_{i,j+1} - \varphi_{i,\frac{j+1}{2}}\right]}{\Delta z_{j+1}} = \frac{D_{i,j}[\varphi_{i,j+1} - \varphi_{i,j}]}{\Delta z_j}$$

$$D_{i,j+1}\Delta z_j \varphi_{i,j+1} - D_{i,j+1}\Delta z_j \varphi_{i,\frac{j+1}{2}} = D_{i,j}\Delta z_{j+1}\varphi_{i,\frac{j+1}{2}} - D_{i,j}\Delta z_{j+1}\varphi_{i,j}$$

$$D_{i,j+1}\Delta z_j \varphi_{i,j+1} + D_{i,j}\Delta z_{j+1}\varphi_{i,j} = D_{i,j+1}\Delta z_j \varphi_{i,\frac{j+1}{2}} + D_{i,j}\Delta z_{j+1}\varphi_{i,\frac{j+1}{2}}$$

$$\varphi_{i,\frac{j+1}{2}} = \frac{D_{i,j+1}\Delta z_j \varphi_{i,j+1} + D_{i,j}\Delta z_{j+1}\varphi_{i,j}}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{j+1}} \qquad 3.25$$

Put 3.25 in 3.23

$$\gamma_{\frac{j+1}{2}} = \frac{2D_{i,j+1}}{\Delta_{i,j+1}}\left[\frac{D_{i,j+1}\Delta z_j \varphi_{i,j+1} + D_{i,j}\Delta z_{j+1}\varphi_{i,j+1} - D_{i,j+1}\Delta z_j \varphi_{i,j+1} - D_{i,j}\Delta z_{j+1}\varphi_{i,j}}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{j+1}}\right]$$

$$\gamma_{\frac{j+1}{2}} = \frac{2D_{i,j+1}}{D_{i,j+1}}\left[\frac{D_{i,j+1}\Delta z_j \varphi_{i,j+1} - D_{i,j}\Delta z_{j+1}\varphi_{i,j}}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{j+1}}\right]$$

$$\gamma_{\frac{j+1}{2}} = \frac{2D_{i,j+1}D_{i,j}}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{j+1}}\left(\varphi_{i,j+1} - \varphi_{i,j}\right) \qquad 3.26$$

$$\gamma_{\frac{j-1}{2}} = \gamma_{\frac{j-1}{2}}+ \quad = \quad \gamma_{\frac{j-1}{2}}- \qquad 3.27$$

$$\gamma_{\frac{j-1}{2}}+= \frac{D_{i,j}[\varphi_{i,j} - \Delta z_{\frac{j+1}{2}}]}{\frac{\Delta z_j}{2}} \qquad 3.28$$

$$\gamma_{\frac{j-1}{2}}-= \frac{D_{i,j-1}\left[\varphi_{i,\frac{j-1}{2}} - \varphi_{i,j-1}\right]}{\Delta z_{\frac{j-1}{2}}} \qquad 3.29$$

$$\frac{2D_{i,j}[\varphi_{i,j} - \varphi_{i,\frac{j-1}{2}}]}{\Delta z_j} = \frac{2D_{i,j-1}[\varphi_{i,\frac{j-1}{2}} - \varphi_{i,j-1}]}{\Delta z_{j-1}}$$

$$D_{i,j}\Delta z_{j-1}\varphi_{i,j} - D_{i,j}\Delta z_{j-1}\varphi_{i,\frac{j-1}{2}} = D_{i,j-1}\Delta z_j\varphi_{i,\frac{j-1}{2}} - D_{i,j-1}\Delta z_j\varphi_{i,j-1}$$

$$D_{i,j}\Delta z_{j-1}\varphi_{i,\frac{j-1}{2}} + D_{i,j-1}\Delta z_j\varphi_{i,\frac{j-1}{2}} = D_{i,j}\Delta z_{j-1}\varphi_{i,j} + D_{i,j-1}\Delta z_j\varphi_{i,j-1}$$

$$\varphi_{i,\frac{j-1}{2}} = \frac{D_{i,j}\Delta z_{j-1}\varphi_{i,j} + D_{i,j-1}\Delta z_j\varphi_{i,j-1}}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_j} \qquad 3.30$$

Put in 3.30 in 3.29

$$\gamma_{\frac{j-1}{2}} = \frac{2D_{i,j-1}}{D_{i,j-1}}\left[\frac{D_{i,j}\Delta z_{j-1}\varphi_{i,j} + D\Delta z_j\varphi_{i,j-1} - D_{i,j}\Delta z_{j-1}\varphi_{i,j-1} - D_{i,j-1}\Delta z_j\varphi_{i,j-1}}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_j}\right]$$

$$\gamma_{\frac{j-1}{2}} = \frac{2D_{i,j-1}}{D_{i,j-1}}\left[\frac{D_{i,j}\Delta z_{j-1}\varphi_{i,j} - D_{i,j}\Delta z_j\varphi_{i,j-1}}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_j}\right]$$

$$\gamma_{\frac{j-1}{2}} = \frac{2D_{i,j-1}}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_j}\left(\varphi_{i,j} - \varphi_{i,j-1}\right) \qquad 3.31$$

Diffusion equation becomes

$$r_{i,j}\left[J_{\frac{i+1}{2}} - J_{\frac{i-1}{2}}\right]\Delta z_j + \left[\gamma_{\frac{j+1}{2}} - \gamma_{\frac{j-1}{2}}\right]\Delta r_i - r_{i,j}\sum\nolimits_{t_{i,j}}\varphi_{i,j}\Delta r\Delta z = -r_{i,j}S_{i,j}\Delta r\Delta z$$

$$\frac{2D_{i+1,j}D_{i,j}r_{i,j}}{D_{i+1,j}\Delta r_i + D_{i,j}\Delta r_{i+1}}\left(\varphi_{i+1,j} - \varphi_{i,j}\right)\Delta z - \frac{2D_{i-1,j}D_{i,j}r_{i,j}\Delta z}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}}\left(\varphi_{i+1,j} - \varphi_{i,j}\right)$$
$$+ \frac{2D_{i,j+1}D_{i,j}r_{i,j}\Delta r}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{j+1}}\left(\varphi_{i,j+1} - \varphi_{i,j}\right)$$
$$- \frac{2D_{i,j-1}D_{i,j}r_{i,j}\Delta r}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_j}\left(\varphi_{i,j} - \varphi_{i,j-1}\right) - r_{i,j}\sum\nolimits_{t_{i,j}}\varphi_{i,j}\Delta r\Delta z$$
$$= -r_{i,j}S_{i,j}\Delta r\Delta z \qquad 3.32$$

$$\propto_{i+1,j} = \frac{2D_{i+1,j}D_{i,j}r_{i,j}}{D_{i+1,j}\Delta r_i + D_{i,j}\Delta r_{i+1}} \qquad 3.33$$

$$\propto_{i-1,j} = \frac{2D_{i-1,j}D_{i,j}r_{i,j}}{D_{i-1,j}\Delta r_i + D_{i,j}\Delta r_{i-1}} \qquad 3.34$$

$$\propto_{i,j+1} = \frac{2D_{i,j+1}D_{i,j}r_{i,j}}{D_{i,j+1}\Delta z_j + D_{i,j}\Delta z_{i+1}} \qquad 3.35$$

$$\propto_{i,j-1} = \frac{2D_{i,j-1}D_{i,j}r_{i,j}}{D_{i,j}\Delta z_{j-1} + D_{i,j-1}\Delta z_i} \qquad 3.36$$

$$\propto_{i+1,j} \left(\varphi_{i+1,j} - \varphi_{i,j}\right)\Delta z - \propto_{i-1,j} \left(\varphi_{i,j} - \varphi_{i-1,j}\right)\Delta z + \propto_{i,j+1} \left(\varphi_{i,j+1} - \varphi_{i,j}\right)\Delta r - \propto_{i,j-1} \left(\varphi_{i,j} - \varphi_{i,j-1}\right)\Delta r - r_{i,j} \sum\nolimits_{t_{i,j}} \varphi_{i,j} \Delta r \Delta z = -r_{i,j} S_{i,j} \Delta r \Delta z$$

$$\begin{aligned}\propto_{i+1,j} \varphi_{i+1,j}\Delta z - \propto_{i+1,j} \varphi_{i,j}\Delta z - \propto_{i-1,j} \varphi_{i,j}\Delta z + \propto_{i-1,j} \varphi_{i-1,j}\Delta z \\ + \propto_{i,j+1} \varphi_{i,j+1}\Delta r - \propto_{i,j+1} \varphi_{i,j}\Delta r - \propto_{i,j-1} \varphi_{i,j}\Delta r \\ + \propto_{i,j-1} \varphi_{i,j-1}\Delta r - r_{i,j} \sum\nolimits_{t_{i,j}} \varphi_{i,j}\Delta r\Delta z = -r_{i,j} S_{i,j}\Delta r\Delta z\end{aligned}$$

Collecting like terms having $\varphi_{i,j}$

$$\begin{aligned}- \propto_{i+1,j} \varphi_{i,j}\Delta z - \propto_{i-1,j} \varphi_{i,j}\Delta z - \propto_{i,j+1} \varphi_{i,j}\Delta r - \propto_{i,j-1} \varphi_{i,j}\Delta r \\ + \propto_{i+1,j} \varphi_{i+1,j}\Delta z + \propto_{i-1,j} \varphi_{i-1,j} \Delta z + \propto_{i,j+1} \varphi_{i,j+1}\Delta r \\ + \propto_{i,j-1} \varphi_{i,j-1}\Delta r - r_{i,j} \sum\nolimits_{t_{i,j}} \varphi_{i,j}\Delta r\Delta z = -r_{i,j} S_{i,j}\Delta r\Delta z\end{aligned}$$

3.37

$$\boldsymbol{\beta_{i,j}\varphi_{i,j}} + \propto_{\boldsymbol{i+1,j}} \boldsymbol{\varphi_{i+1,j}} + \propto_{\boldsymbol{i-1,j}} \boldsymbol{\varphi_{i-1,j}} + \propto_{\boldsymbol{i,j+1}} \boldsymbol{\varphi_{i,j+1}} + \propto_{\boldsymbol{i,j-1}} \boldsymbol{\varphi_{i,j-1}} = \boldsymbol{F_{i,j}} \qquad 3.38$$

$$\boldsymbol{\beta_{i,j}} = -\left[\propto_{\boldsymbol{i+1,j}} + \propto_{\boldsymbol{i-1,j}} + \propto_{\boldsymbol{i,j+1}} + \propto_{\boldsymbol{i,j-1}} + r_{i,j} \sum\nolimits_{t_{i,j}} \varphi_{i,j}\Delta r\Delta z\right] \qquad 3.39$$

$$\boldsymbol{F_{i,j}} = -r_{i,j} S_{i,j}\Delta r\Delta z \qquad 3.40$$

Equation (3.38) is the five point difference equation for the computation of the neutron flux of any energy group g in a finite cylindrical reactor.