Chem 485 PS4 Solutions

1) $A(x,y,z) = \frac{E_0}{z-iz_0} \exp\left(ik \frac{x^2+y^2}{2(z-iz_0)}\right)$

$$\frac{\partial A}{\partial z} = -\left[\frac{1}{z-iz_0} + \frac{ik(x^2+y^2)}{2(z-iz_0)}\right] A$$

Compare the magnitude of this quantity to kA:

$$\left|\frac{1}{z-iz_0} + \frac{ik(x^2+y^2)}{2(z-iz_0)^2}\right| A \;\underset{\uparrow}{??}\; kA$$

$>, <, =, \gg, \ll$, etc?

For the paraxial beam, begin by comparing these quantities on the optical axis $(x=y=0)$. The term on the left is $=0$ for $z \to \pm\infty$, and has its largest value at $z=0$:

$$\left|\frac{1}{z_0}\right| \;??\; k$$

divide both sides by k

$$\frac{1}{k}\left|\frac{1}{z_0}\right| = \frac{1}{2}\left(\frac{\lambda}{\pi w_0}\right)^2 = \frac{1}{2}\theta_0^2 \ll 1$$

The assumption is valid on the optical axis. Returning to the term in $\frac{\partial A}{\partial z}$ off axis:

$$\frac{\cancel{k}(x^2+y^2)}{2(z-iz_0)} \;??\; \cancel{k}$$

But x^2+y^2 is of the order w^2 (i.e. A has a significantly large value within this range), so compare

$$\frac{w^2}{2(z-iz_0)^2} \;??\; 1$$

$$w^2 \cdot \frac{1}{2(z-iz_0)^2} = \frac{\lambda}{\pi}\left(\frac{z_0^2+z^2}{z_0}\right)\frac{1}{2(z-iz_0)^2} = \frac{\lambda}{2\pi z_0}\underbrace{\left(\frac{z+iz_0}{z-iz_0}\right)}_{\text{order } 1}$$

$$= \frac{1}{2}\left(\frac{\lambda}{\pi w_0}\right)^2 = \frac{1}{2}\theta_0^2 \ll 1$$

Both terms in $\frac{\partial A}{\partial z}$ are negligible compared to kA when $\theta_0 < 1$.

2) a) $R = \frac{z^2 - z_0^2}{z}$

$z_0 = \sqrt{zR - z^2}$, $z^2 + z_0^2 = zR$

$w^2 = w_0^2\left(1 + \left(\frac{z}{z_0}\right)^2\right)$

$= \frac{\lambda}{\pi} \frac{z_0^2 + z^2}{z_0}$

$$zR - z^2 = \left(\frac{\lambda R}{\pi w^2}\right)^2 z^2$$

$$z = \frac{R}{1 + \left(\frac{\lambda R}{\pi w^2}\right)^2}$$

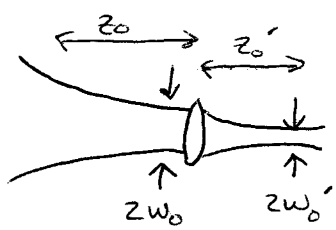
$$w_0 = w\left[\frac{z_0^2}{z^2 + z_0^2}\right]^{1/2}$$

$$= w\left[\frac{zR - z^2}{zR}\right]^{1/2}$$

$$= w\left[1 - \frac{1}{1 + \left(\frac{\lambda R}{\pi w^2}\right)^2}\right]^{1/2}$$

$$= \frac{w}{\left[1 + \left(\pi w^2/\lambda R\right)^2\right]^{1/2}}$$

b)



just before the lens, $R_0 = \infty$, so just after the lens, R_0' is:

$$\frac{1}{R_0'} = \frac{1}{\infty} - \frac{1}{f} \Rightarrow R_0' = -f \quad + \quad w = w_0'$$

$$\therefore \quad z' = \frac{f}{1 + \left(\frac{\lambda f}{\pi w_0^2}\right)^2} = \frac{f}{1 + (f/z_0)^2}$$

where $z_0 = \frac{\pi w_0^2}{\lambda}$

$$w_0' = \frac{w_0}{\left[1 + \left(\pi w_0^2/\lambda f\right)^2\right]^{1/2}} = \frac{w_0}{\left[1 + (z_0/f)^2\right]^{1/2}}$$

c) for $z_0 \rightarrow$ large:

$$z' = f$$

$$w_0' = \frac{w_0}{z_0} = \frac{\lambda}{\pi w_0}$$

3) a) $q' = \dfrac{Aq+B}{Cq+D}$

$\#1 \rightarrow q' = q \Rightarrow \quad q = \dfrac{Aq+B}{Cq+D}$

$$Cq^2 + (D-A)q - B = 0$$

$$q = \frac{-(D-A) \pm \sqrt{(D-A)^2 + 4CB}}{2C}$$

$\#2 \rightarrow (D-A)^2 + 4CB < 0$ (this is the condition for q to have an imaginary part)

$\#3 \rightarrow CB = -1 + AD$

$$D^2 - 2AD + A^2 + 4AD - 4 < 0$$

$$(D+A)^2 - 4 < 0 \quad \text{or} \quad -2 < D+A < 2$$

b) for the resonator pictured, a round trip consists of

$$M = \begin{pmatrix} 1 & 0 \\ -2/R_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & d \\ -2/R_1 & 1 - \frac{2d}{R_1} \end{pmatrix}$$

$$\begin{pmatrix} 1 - \frac{2d}{R_1} & 2d - \frac{2d^2}{R_1} \\ -2/R_1 & 1 - \frac{2d}{R_1} \end{pmatrix}$$

$$\begin{pmatrix} 1 - \frac{2d}{R_1} & 2d - \frac{2d^2}{R_1} \\ -\frac{2}{R_2} + \frac{4d}{R_1 R_2} - \frac{2}{R_1} & \frac{-4d}{R_2} + \frac{4d^2}{R_1 R_2} + 1 - \frac{2d}{R_1} \end{pmatrix}$$

3) b (continued)

for a stable resonator (from part (a)) we need:

$$-2 < D + A < 2$$

$$-2 < 2 - \frac{4d}{R_1} - \frac{4d}{R_2} + \frac{4d^2}{R_1R_2} < 2$$

$$0 < 4\left(1 - \frac{d}{R_1} - \frac{d}{R_2} + \frac{d^2}{R_1R_2}\right) < 4$$

$$0 < \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) < 1$$

4) power $\propto |E|^2$

We calculate the ratio of the power transmitted through a circular aperture of radius w using cyllindrical coordinates at a point along the optical axis.

$$T = \frac{\int_{r=0}^{w}\int_{\theta=0}^{2\pi} |E|^2 r\, d\theta\, dr}{\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} |E|^2 r\, d\theta\, dr}$$

$TEM_{00} \rightarrow$ since $x^2+y^2=r^2$, $|E|^2 \propto e^{-2r^2/w^2}$ (angular integrals cancel)

$$T = \frac{\int_{r=0}^{w} r e^{-2r^2/w^2} dr}{\int_{r=0}^{\infty} r e^{-2r^2/w^2} dr} = \frac{\left[-\frac{w^2}{4} e^{-2r^2/w^2}\right]_0^w}{\left[-\frac{w^2}{4} e^{-2r^2/w^2}\right]\Big|_0^\infty} = 1-e^{-2} \simeq 0.86$$

$TEM_{10} = TEM_{01}$ (by symmetry)

Again, angular integrals will cancel, $x^2+y^2=r^2$ and $x=r\cos\theta$:

$$T = \frac{\int_{r=0}^{w} r^3 e^{-2r^2/w^2}}{\int_{r=0}^{\infty} r^3 e^{-2r^2/w^2}} = \frac{\left[-e^{-2r^2/w^2}\left(\frac{2r^2}{w^2}+1\right)\right]_0^w}{\left[-e^{-2r^2/w^2}\left(\frac{2r^2}{w^2}+1\right)\right]\Big|_0^\infty} = 1-3e^{-2} \simeq 0.59$$

TEM_{11}

Angular integrals will again cancel, but $|E| \propto x \cdot y \exp(\;) \propto r^2 \exp(\;)$:

$$T = \frac{\int_{r=0}^{w} r^5 e^{-2r^2/w^2}}{\int_{r=0}^{\infty} r^5 e^{-2r^2/w^2}} = \frac{\left[-e^{-2r^2/w^2}\left(\frac{4r^4}{w^4}+\frac{4r^2}{w^2}+2\right)\right]\Big|_0^w}{\left[-e^{-2r^2/w^2}\left(\frac{4r^4}{w^4}+\frac{4r^2}{w^2}+2\right)\right]\Big|_0^\infty} = 1-5e^{-2} \simeq 0.32$$

4 (ctd) If we had considered an aperture of $\frac{w}{3}$, the transmission would have been:

$$TEM_{00} : T = 1 - e^{-2/9} = 0.199$$
$$TEM_{11} : T = 1 - e^{-2/9}\left(\frac{2}{3^4} + \frac{2}{3^2} + 1\right) = 1.55 \times 10^{-3}$$

The discrimination is much larger $\left(\frac{TEM_{00}}{TEM_{10}} \sim 128\right)$ for $\frac{w}{3}$ than for an aperture of w $\left(\frac{TEM_{00}}{TEM_{11}} \sim 2.7\right)$, but much less power is transmitted overall.