The problem: Let $f(t)$ be analytic at $t = 0$ with $f(0) = 0$ and $f'(0) \neq 0$. Let C be a circle centred on the origin, with interior D, such that f is analytic in D and the inverse of f exists on $f(D)$. For a fixed point z within C, let $w = f(z)$. Assuming that w is small, show using the residue theorem that

$$z = \frac{1}{2\pi i} \int_C \frac{t f'(t)}{f(t) - w} dt$$

Attempt at solution: The function $\frac{tf'(t)}{f(t)-w}$ has a simple pole at $t = z$ with residue

$$Res(z) = \lim_{t \to z} t f'(t) = z f'(z)$$

Therefore by the residue theorem

$$\frac{1}{2\pi i} \int_C \frac{t f'(t)}{f(t) - w} dt = \frac{1}{2\pi i} \times 2\pi i (z f'(z)) = z f'(z)$$