Conformal Transformation of the Ricci Scalar

The condition for a transformation to be a conformal transformation

$$\begin{aligned} g_{ab} &\to \Omega^2 g_{ab} \\ g^{ab} &\to \Omega^{-2} g^{ab}, \end{aligned}$$

implies that the determinant of the metric transforms as

$$g = \varepsilon^{\mu_0 .. \mu_{D-1}} g_{0\mu_0} .. g_{(D-1)\mu_{(D-1)}} \to \Omega^{2D} g$$

so that the volume element transforms as

$$\sqrt{-g} \to \Omega^D \sqrt{-g}.$$

and we note the partial derivative of the metric is

$$\partial_a g_{bc} = \Gamma_{b\,ac} + \Gamma_{c\,ba}$$

where the Christoffel symbol is defined by

$$\begin{aligned} \Gamma^a{}_{bc} &= g^{ad}\Gamma_{d\,bc} \\ &= \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \end{aligned}$$

under a conformal transformation it transforms as

$$\begin{aligned} \Gamma^a{}_{bc} \to \Gamma^{a\prime}{}_{bc} &= \frac{1}{2}\Omega^{-2} g^{ad}[\partial_b(\Omega^2 g_{cd}) + \partial_c(\Omega^2 g_{bd}) - \partial_d(\Omega^2 g_{bc})] \\ &= \Gamma^a{}_{bc} + \frac{1}{2}\Omega^{-2}[g^{ad} g_{cd}\partial_b\Omega^2 + g^{ad} g_{bd}\partial_c\Omega^2 - g^{ad} g_{bc}\partial_d\Omega^2] \\ &= \Gamma^a{}_{bc} + \Omega^{-1}[\delta^a{}_c\partial_b\Omega + \delta^a{}_b\partial_c\Omega - g_{bc}\partial^a\Omega]. \end{aligned}$$

We can now simply use the conformal transformation of the Christoffel symbols to compute the conformal transformation of the Riemann tensor $R^a{}_{bcd}$ defined such that the a is upper and the b is the last index in each term if we can't contract the last index with another (non-derivative) term

$$\begin{aligned} R^a{}_{bcd} &= 2\partial_{[c}\Gamma^a{}_{d]b} + 2\Gamma^a{}_{[c|p|}\Gamma^p{}_{d]b} \\ &= \partial_c\Gamma^a{}_{db} - \partial_d\Gamma^a{}_{cb} + \Gamma^a{}_{cp}\Gamma^p{}_{db} - \Gamma^a{}_{dp}\Gamma^p{}_{cb} \end{aligned}$$

[note we contracted the last index in the first term of the (anti-symmetric) product of Christoffel's via p] directly

$$
\begin{aligned}
R^{a\prime}{}_{bcd} &= 2\partial_{[c}\Gamma^{a\prime}{}_{d]b} + 2\Gamma^{a\prime}{}_{[c|p|}\Gamma^{p\prime}{}_{d]b} \\
&= 2\partial_{[c}\{\Gamma^{a}{}_{d]b} + \Omega^{-1}(\delta^{a}{}_{|b|}\partial_{d]}\Omega + \delta^{a}{}_{d]}\partial_b\Omega - g_{d]b}\partial^a\Omega)\} \\
&+ 2\Gamma^{a\prime}{}_{[c|p|}\Gamma^{p\prime}{}_{d]b} \\
&= 2\partial_{[c}\{\Gamma^{a}{}_{d]b} + \Omega^{-1}(\delta^{a}{}_{|b|}\partial_{d]}\Omega + \delta^{a}{}_{d]}\partial_b\Omega - g_{d]b}\partial^a\Omega)\} \\
&+ 2\{\Gamma^{a}{}_{[c|p|} + \Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\}\{\Gamma^{p}{}_{d]b} + \Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&= 2\partial_{[c}\Gamma^{a}{}_{d]b} + 2\partial_{[c}\{\Omega^{-1}(\delta^{a}{}_{|b|}\partial_{d]}\Omega + \delta^{a}{}_{d]}\partial_b\Omega - g_{d]b}\partial^a\Omega)\} \\
&+ 2\Gamma^{a}{}_{[c|p|}\{\Gamma^{p}{}_{d]b} + \Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&+ 2\Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\{\Gamma^{p}{}_{d]b} + \Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&= 2\partial_{[c}\Gamma^{a}{}_{d]b} + 2\partial_{[c}\{\Omega^{-1}(\delta^{a}{}_{|b|}\partial_{d]}\Omega + \delta^{a}{}_{d]}\partial_b\Omega - g_{d]b}\partial^a\Omega)\} \\
&+ 2\Gamma^{a}{}_{[c|p|}\Gamma^{p}{}_{d]b} + 2\Gamma^{a}{}_{[c|p|}\{\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&+ 2\Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\Gamma^{p}{}_{d]b} \\
&+ 2\Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\{\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&= R^{a}{}_{bcd} \\
&+ 2\partial_{[c}\{\Omega^{-1}(\delta^{a}{}_{|b|}\partial_{d]}\Omega + \delta^{a}{}_{d]}\partial_b\Omega - g_{d]b}\partial^a\Omega)\} \\
&+ 2\Gamma^{a}{}_{[c|p|}\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \\
&+ 2\Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\Gamma^{p}{}_{d]b} \\
&+ 2\Omega^{-1}[\delta^{a}{}_{p}\partial_{[c}\Omega + \delta^{a}{}_{[c}\partial_{|p|}\Omega - g_{[c|p|}\partial^a\Omega]\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]
\end{aligned}
$$

We could simplify further, but our goal is to compute the conformal transformation of the Ricci scalar

$$R = g^{bd}R^{x}{}_{bxd},$$

and since the Ricci scalar involves g^{-1} we expect each term to be weighted by Ω^{-2}, and since the Ricci scalar involves two derivatives we expect it to schematically take the form

$$R = g^{bd}R^{x}{}_{bxd} \;\to\; R' = \Omega^{-2}R + \lambda_1\Omega^{-3}g^{ab}D_a\partial_b\Omega + \lambda_2\Omega^{-4}g^{ab}(\partial_a\Omega)(\partial_b\Omega)$$

We fix the coefficients λ_1, λ_2 by contracting indices in the Ricci tensor to get the Ricci scalar

$$
\begin{aligned}
R = g^{bd}R^{x}{}_{bxd} \to R' &= \Omega^{-2}g^{bd}R^{x\prime}{}_{bxd} \\
&= \Omega^{-2}g^{bd}\{R^{a}{}_{bxd} \\
&+ 2\partial_{[x}\{\Omega^{-1}(\delta^{x}{}_{|b|}\partial_{d]}\Omega + \delta^{x}{}_{d]}\partial_b\Omega - g_{d]b}\partial^x\Omega)\} \\
&+ 2\Gamma^{x}{}_{[x|p|}\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \\
&+ 2\Omega^{-1}[\delta^{x}{}_{p}\partial_{[x}\Omega + \delta^{x}{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega]\Gamma^{p}{}_{d]b} \\
&+ 2\Omega^{-1}[\delta^{x}{}_{p}\partial_{[x}\Omega + \delta^{x}{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega]\Omega^{-1}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega]\} \\
&= \Omega^{-2}R \\
&- \Omega^{-4}g^{bd}2\{\partial_{[x}\Omega\}(\delta^{x}{}_{|b|}\partial_{d]}\Omega + \delta^{x}{}_{d]}\partial_b\Omega - g_{d]b}\partial^x\Omega) \quad I \\
&\qquad + \Omega^{-3}g^{bd}2\partial_{[x}\{(\delta^{x}{}_{|b|}\partial_{d]}\Omega + \delta^{x}{}_{d]}\partial_b\Omega - g_{d]b}\partial^x\Omega)\} \quad II \\
&+ 2\Gamma^{x}{}_{[x|p|}\Omega^{-3}g^{bd}[\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \quad III \\
&+ \Omega^{-3}g^{bd}2[\delta^{x}{}_{p}\partial_{[x}\Omega + \delta^{x}{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega]\Gamma^{p}{}_{d]b} \quad IV \\
&+ \Omega^{-4}g^{bd}2[\delta^{x}{}_{p}\partial_{[x}\Omega + \delta^{x}{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega][\delta^{p}{}_{|b|}\partial_{d]}\Omega + \delta^{p}{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \quad V.
\end{aligned}
$$

We now evaluate each non-trivial line, first evaluating I

$$\begin{aligned}
I &= -2\Omega^{-4} g^{bd} \{\partial_{[x}\Omega\} (\delta^x{}_{|b|}\partial_{d]}\Omega + \delta^x{}_{d]}\partial_b\Omega - g_{d]b}\partial^x\Omega) \\
&= -\Omega^{-4} g^{bd}\{\partial_x\Omega\}(\delta^x{}_b\partial_d\Omega + \delta^x{}_d\partial_b\Omega - g_{db}\partial^x\Omega) + \Omega^{-4}g^{bd}\{\partial_d\Omega\}(\delta^x{}_b\partial_x\Omega + \delta^x{}_x\partial_b\Omega - g_{xb}\partial^x\Omega) \\
&= -\Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega - \Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega + D\Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega + \Omega^{-4}\{\partial^x\Omega\}\partial_x\Omega + D\Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega - \Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega \\
&= 2(D-1)\Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega
\end{aligned}$$

then evaluating II

$$\begin{aligned}
II &= \Omega^{-3} 2 g^{bd}\partial_{[x}\{(\delta^x{}_{|b|}\partial_{d]}\Omega + \delta^x{}_{d]}\partial_b\Omega - g_{d]b}\partial^x\Omega)\} \\
&= \Omega^{-3}g^{bd}\partial_x\{(\delta^x{}_b\partial_d\Omega + \delta^x{}_d\partial_b\Omega - g_{db}\partial^x\Omega)\} - \Omega^{-3}g^{bd}\partial_d\{(\delta^x{}_b\partial_x\Omega + \delta^x{}_x\partial_b\Omega - g_{xb}\partial^x\Omega)\} \\
&= \Omega^{-3}g^{bd}\partial_b\partial_d\Omega + \Omega^{-3}g^{bd}\partial_d\partial_b\Omega - \Omega^{-3}g^{bd}\partial_x[g_{db}\partial^x\Omega] - \Omega^{-3}g^{bd}\partial_d\partial_b\Omega - D\Omega^{-3}g^{bd}\partial_d\partial_b\Omega + \Omega^{-3}g^{bd}\partial_d\partial_b\Omega \\
&= -(D-2)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}\partial_x[g_{db}\partial^x\Omega] \\
&= -(D-2)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}\partial_x[g_{db}g^{xe}\partial_e\Omega] \\
&= -(D-2)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}[\partial_x g_{db}]g^{xe}\partial_e\Omega - \Omega^{-3}g^{bd}g_{db}[\partial_x g^{xe}]\partial_e\Omega - \Omega^{-3}g^{bd}g_{db}g^{xe}\partial_x\partial_e\Omega \\
&= -(D-2)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}[\partial_x g_{db}]g^{xe}\partial_e\Omega - \Omega^{-3}D[\partial_x g^{xe}]\partial_e\Omega - D\Omega^{-3}g^{xe}\partial_x\partial_e\Omega \\
&= -2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}[\partial_x g_{db}]g^{xe}\partial_e\Omega - \Omega^{-3}D[\partial_x g^{xe}]\partial_e\Omega \\
&= -2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}[\Gamma_{d\,xb} + \Gamma_{b\,dx}]\partial^x\Omega - \Omega^{-3}D[\partial_x g^{xe}]\partial_e\Omega \\
&= -2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}g^{bd}[\Gamma_{d\,xb} + \Gamma_{b\,dx}]\partial^x\Omega + \Omega^{-3}D[\Gamma^x{}_x{}^e + \Gamma^{ex}{}_x]\partial_e\Omega \\
&= -2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - \Omega^{-3}\Gamma^b{}_{xb}\partial^x\Omega - \Omega^{-3}\Gamma^d{}_{dx}\partial^x\Omega + D\Omega^{-3}\Gamma^x{}_x{}^e\partial_e\Omega + D\Omega^{-3}\Gamma^{ex}{}_x\partial_e\Omega \\
&= -2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - 2\Omega^{-3}\Gamma^b{}_{xb}\partial^x\Omega + D\Omega^{-3}\Gamma^x{}_x{}^e\partial_e\Omega + D\Omega^{-3}\Gamma^{ex}{}_x\partial_e\Omega,
\end{aligned}$$

where we used

$$\begin{aligned}
g_{ab}b^{bc} &= \delta_a{}^c \\
\partial_d g_{ab} &= \Gamma_{a\,db} + \Gamma_{b\,ad} \\
0 &= \partial_d(g_{ab}g^{bc}) \\
&= (\partial_d g_{ab})g^{bc} + g_{ab}(\partial_d g^{bc}) = (\Gamma_{a\,db} + \Gamma_{b\,ad})g^{bc} + g_{ab}(-\Gamma^b{}_d{}^c - \Gamma^{cb}{}_d) = (\Gamma_{a\,db} + \Gamma_{b\,ad})g^{bc} - (\Gamma_{ad}{}^c + \Gamma^c{}_{ad}) \\
&= (\Gamma_{a\,db} + \Gamma_{b\,ad})g^{bc} - (\Gamma_{a db} + \Gamma_{b\,ad})g^{bc} \\
\partial_d g^{bc} &= -(\Gamma^b{}_d{}^c + \Gamma^{cb}{}_d)
\end{aligned}$$

then evaluating III

$$\begin{aligned}
III &= \Omega^{-3}2g^{bd}\Gamma^x{}_{[x|p|}[\delta^p{}_{|b|}\partial_{d]}\Omega + \delta^p{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \\
&= \Omega^{-3}g^{bd}\Gamma^x{}_{xp}[\delta^p{}_b\partial_d\Omega + \delta^p{}_d\partial_b\Omega - g_{db}\partial^p\Omega] - \Omega^{-3}g^{bd}\Gamma^x{}_{dp}[\delta^p{}_b\partial_x\Omega + \delta^p{}_x\partial_b\Omega - g_{xb}\partial^p\Omega] \\
&= \Omega^{-3}\Gamma^x{}_{xp}[\partial^p\Omega + \partial^p\Omega - D\partial^p\Omega] - \Omega^{-3}\Gamma^x{}_{dp}[g^{pd}\partial_x\Omega + \delta^p{}_x\partial^d\Omega - \delta_x{}^d\partial^p\Omega] \\
&= (2-D)\Omega^{-3}\Gamma^x{}_{xp}\partial^p\Omega - \Omega^{-3}\Gamma^x{}_d{}^d\partial_x\Omega - \Omega^{-3}\Gamma^x{}_{xp}\partial^p\Omega + \Omega^{-3}\Gamma^x{}_{xp}\partial^p\Omega \\
&= (2-D)\Omega^{-3}\Gamma^x{}_{xp}\partial^p\Omega - \Omega^{-3}\Gamma^x{}_d{}^d\partial_x\Omega
\end{aligned}$$

then evaluating IV

$$\begin{aligned}
IV &= \Omega^{-3}2g^{bd}[\delta^x{}_p\partial_{[x}\Omega + \delta^x{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega]\Gamma^p{}_{d]b} \\
&= \Omega^{-3}g^{bd}[\delta^x{}_p\partial_x\Omega + \delta^x{}_x\partial_p\Omega - g_{xp}\partial^x\Omega]\Gamma^p{}_{db} - \Omega^{-3}g^{bd}[\delta^x{}_p\partial_d\Omega + \delta^x{}_d\partial_p\Omega - g_{dp}\partial^x\Omega]\Gamma^p{}_{xb} \\
&= \Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d + D\Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d - \Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d - \Omega^{-3}\partial_d\Omega\Gamma^p{}_p{}^d - \Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d + \Omega^{-3}\partial^x\Omega\Gamma^p{}_{xp} \\
&= D\Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d - \Omega^{-3}\partial^d\Omega\Gamma^p{}_{pd} - \Omega^{-3}\partial_p\Omega\Gamma^p{}_d{}^d + \Omega^{-3}\partial^d\Omega\Gamma^p{}_{pd} \\
&= -(1-D)\Omega^{-3}\partial_x\Omega\Gamma^x{}_d{}^d
\end{aligned}$$

then evaluating V

$$
\begin{aligned}
V &= \Omega^{-4} 2 g^{bd} [\delta^x{}_p \partial_{[x}\Omega + \delta^x{}_{[x}\partial_{|p|}\Omega - g_{[x|p|}\partial^x\Omega][\delta^p{}_{|b|}\partial_{d]}\Omega + \delta^p{}_{d]}\partial_b\Omega - g_{d]b}\partial^p\Omega] \\
&= \Omega^{-4} g^{bd} [\delta^x{}_p \partial_x\Omega + \delta^x{}_x\partial_p\Omega - g_{xp}\partial^x\Omega][\delta^p{}_b\partial_d\Omega + \delta^p{}_d\partial_b\Omega - g_{db}\partial^p\Omega] \\
&\quad - \Omega^{-4} g^{bd} [\delta^x{}_p \partial_d\Omega + \delta^x{}_d\partial_p\Omega - g_{dp}\partial^x\Omega][\delta^p{}_b\partial_x\Omega + \delta^p{}_x\partial_b\Omega - g_{xb}\partial^p\Omega] \\
&= \Omega^{-4} [\delta^x{}_p \partial_x\Omega + \delta^x{}_x\partial_p\Omega - \partial_p\Omega][\partial^p\Omega + \partial^p\Omega - D\partial^p\Omega] \\
&\quad - \Omega^{-4} [\delta^x{}_p \partial_d\Omega + \delta^x{}_d\partial_p\Omega - g_{dp}\partial^x\Omega][g^{pd}\partial_x\Omega + \delta^p{}_x\partial^d\Omega - \delta^d{}_x\partial^p\Omega] \\
&= -(D-2)\Omega^{-4}[\partial_p\Omega\partial^p\Omega + D\partial_p\Omega\partial^p\Omega - \partial_p\Omega\partial^p\Omega] \\
&\quad - \Omega^{-4}\{\delta^x{}_p\partial_d\Omega[g^{pd}\partial_x\Omega + \delta^p{}_x\partial^d\Omega - \delta^d{}_x\partial^p\Omega] + \delta^x{}_d\partial_p\Omega[g^{pd}\partial_x\Omega + \delta^p{}_x\partial^d\Omega - \delta^d{}_x\partial^p\Omega] \\
&\qquad\qquad - g_{dp}\partial^x\Omega[g^{pd}\partial_x\Omega + \delta^p{}_x\partial^d\Omega - \delta^d{}_x\partial^p\Omega]\} \\
&= -D(D-2)\Omega^{-4}\partial_p\Omega\partial^p\Omega \\
&\quad - \Omega^{-4}\{\partial_x\Omega[\partial^x\Omega + D\partial^x\Omega - \partial^x\Omega] + \partial_x\Omega[\partial^x\Omega + \partial^x\Omega - D\partial^x\Omega] \\
&\qquad\qquad - \partial^x\Omega[D\partial_x\Omega + \partial_x\Omega - \partial_x\Omega]\} \\
&= -D(D-2)\Omega^{-4}\partial_p\Omega\partial^p\Omega - \Omega^{-4}\{D\partial_x\Omega\partial^x\Omega + (2-D)\partial_x\Omega\partial^x\Omega - D\partial_x\Omega\partial^x\Omega\} \\
&= -D(D-2)\Omega^{-4}\partial_p\Omega\partial^p\Omega - \Omega^{-4}(2-D)\partial_x\Omega\partial^x\Omega.
\end{aligned}
$$

This results in

$$
\begin{aligned}
R' &= \Omega^{-2}R + I + \ldots + V \\
&= \Omega^{-2}R \\
&+ 2(D-1)\Omega^{-4}\{\partial_x\Omega\}\partial^x\Omega \\
&- 2(D-1)\Omega^{-3}g^{bd}\partial_b\partial_d\Omega - 2\Omega^{-3}\Gamma^b{}_{xb}\partial^x\Omega + D\Omega^{-3}\Gamma^x{}_x{}^e\partial_e\Omega + D\Omega^{-3}\Gamma^{ex}{}_x\partial_e\Omega \\
&+ (2-D)\Omega^{-3}\Gamma^x{}_{xp}\partial^p\Omega - \Omega^{-3}\Gamma^x{}_d{}^d\partial_x\Omega \\
&- (1-D)\Omega^{-3}\partial_x\Omega\Gamma^x{}_d{}^d \\
&- D(D-2)\Omega^{-4}\partial_p\Omega\partial^p\Omega - \Omega^{-4}(2-D)\partial_x\Omega\partial^x\Omega \\
&= \Omega^{-2}R \\
&+ \Omega^{-3}\{-2(D-1)g^{bd}\partial_b\partial_d\Omega - 2\Gamma^b{}_{xb}\partial^x\Omega + D\Gamma^x{}_x{}^e\partial_e\Omega + D\Gamma^{ex}{}_x\partial_e\Omega + (2-D)\Gamma^x{}_{xp}\partial^p\Omega - \Gamma^x{}_d{}^d\partial_x\Omega \\
&\qquad\qquad - (1-D)\partial_x\Omega\Gamma^x{}_d{}^d\} \\
&+ \Omega^{-4}\{2(D-1) - D(D-2) - (2-D)\}\partial_x\Omega\partial^x\Omega \\
&= \Omega^{-2}R \\
&+ \Omega^{-3}\{-2(D-1)g^{bd}\partial_b\partial_d\Omega + D\Gamma^{ex}{}_x\partial_e\Omega - \Gamma^x{}_d{}^d\partial_x\Omega - (1-D)\partial_x\Omega\Gamma^x{}_d{}^d\} \\
&+ \Omega^{-4}\{2(D-1) - D(D-2) + 1(D-2)\}\partial_x\Omega\partial^x\Omega \\
&= \Omega^{-2}R \\
&+ \Omega^{-3}\{-2(D-1)g^{bd}\partial_b\partial_d\Omega + D\Gamma^{ex}{}_x\partial_e\Omega - (2-D)\Gamma^x{}_d{}^d\partial_x\Omega\} \\
&+ \Omega^{-4}\{2(D-1) - (D-1)(D-2)\}\partial_x\Omega\partial^x\Omega \\
&= \Omega^{-2}R + g^{bd}\{-2(D-1)\partial_b\partial_d\Omega + 2(D-1)\Gamma^x{}_d{}^d\partial_x\Omega\}\frac{1}{\Omega^3} - (D-1)(D-4)\frac{\partial_x\Omega\partial^x\Omega}{\Omega^4} \\
&= \Omega^{-2}R - 2(D-1)g^{bd}\{\partial_b\partial_d\Omega - \Gamma^c{}_{bd}\partial_c\Omega\}\frac{1}{\Omega^3} - (D-1)(D-4)\frac{\partial_x\Omega\partial^x\Omega}{\Omega^4} \\
&= \Omega^{-2}R - 2(D-1)g^{bd}D_b\partial_d\Omega\frac{1}{\Omega^3} - (D-1)(D-4)\frac{\partial_x\Omega\partial^x\Omega}{\Omega^4} \\
&= \Omega^{-2}R - 2(D-1)g^{bd}D_bD_d\Omega\frac{1}{\Omega^3} - (D-1)(D-4)\frac{\partial_x\Omega\partial^x\Omega}{\Omega^4}.
\end{aligned}
$$

The conformal transformation of the Ricci tensor is

$$
R'_{bd} = R_{bd} - [(D-2)\delta^x{}_b\delta^y{}_d + g_{bd}g^{xy}]\frac{D_x\partial_y\Omega}{\Omega} + [2(D-2)\delta^x{}_b\delta^y{}_d - (D-3)g_{bd}g^{xy}]\frac{(\partial_x\Omega)(\partial_y\Omega)}{\Omega^2}
$$