2.
The proof for part two is similar to that of part 1.

Since $\lim_{x \to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.
Also, since
$\lim_{x \to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:

$[f(x) - g(x)] - [A - B] = \gamma(x) \quad ???$. Where $\gamma(x)$ is also an infinitesimal. In other words we need to show that the difference $[f(x) - g(x)] - [A - B]$ is actually an infinitesimal. Let that infinitesimal be any $\gamma(x)$

So,

$[f(x) - g(x)] - [A - B] = [f(x) - A] - [g(x) - B] = \alpha(x) - \beta(x) = \gamma(x)$. We know from e previous theorem that the difference of two infinitesimals is again an infinitesimal.
So we proved this part of the theorem too.

3.

Since $\lim_{x \to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.
Also, since
$\lim_{x \to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:

$[f(x) * g(x)] - [A * B] = \gamma(x) \quad ???$

Where $\gamma(x)$ is an arbitrary infinitesimal.

So,

$$[f(x)*g(x)]-[A*B] = f(x)*g(x)-g(x)*A+g(x)*A-A*B = g(x)[f(x)-A]+A[g(x)-B] = g(x)*\alpha(x)+A*\beta(x) = \alpha'(x)+\beta'(x) = \gamma(x).$$

From the above theorems we know that the product of any infinitesimal with a function that has limit at a point is also an infinitesimal.($g(x) * \alpha(x) = \alpha'(x)$), also the product of a constant and an infinitesimal is also an infinitesimal ($A * \beta(x) = \beta'(x)$.)

This way we have proved part 3 of the theorem.

4.

Since $\lim_{x \to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.

$$[\frac{f(x)}{g(x)}] - \frac{A}{B} = \frac{Bf(x) - Ag(x)}{Bg(x)} = \frac{Bf(x) - AB + AB - Ag(x)}{Bg(x)} = \frac{B[f(x) - A] - A[g(x) - B]}{Bg(x)} = \frac{B\alpha(x) - A\beta(x)}{Bg(x)}$$

Also, since
$\lim_{x \to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:

$$[\frac{f(x)}{g(x)}] - \frac{A}{B} = \gamma(x) \quad ????$$

Where $\gamma(x)$ is an arbitrary infinitesimal.

This way:

Now using the results of the above theorems and corollaries we get:

$$\frac{B\alpha(x) - A\beta(x)}{Bg(x)} =^{1)} \frac{\alpha'(x)}{Bg(x)} = \frac{1}{B} * \frac{\alpha'(x)}{g(x)} =^{2)} \frac{1}{B} * \beta'(x) =^{3)} \gamma(x)$$

1) From a theorem before we know that any linear combination of the infinitesimals is still an infinitesimal: $B\alpha(x) - A\beta(x) = \alpha'(x)$, where $\alpha'(x)$ is also an infinitesimal.

2) As a result of Corollary 1.
3) Corollary 3.

This way the part 4 of the theorem is proved.

5. The proof for this part is rather trivial. However, I am going to prove it here again.

Since $\lim_{x\to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.
Also, since
$\lim_{x\to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:
$cf(x) - cA = \gamma(x)$?????

Where $\gamma(x)$ is an arbitrary infinitesimal.

So: $cf(x) - cA = c(f(x) - A) = c\,\alpha(x) =^{1)} \gamma(x)$

1)As a consequence of corollary 3.

This way part 5 is proved.

6. I will approach this proof in a slightly different manner than the other ones.

Since $\lim_{x\to a} f(x) = A$, it actually means that $f(x) - A = \alpha(x)$.
Also, since
$\lim_{x\to a} g(x) = B$, from the definition it means that $g(x) - B = \beta(x)$. Where both $\alpha(x) \quad and \quad \beta(x)$ are infinitesimals.

In order to prove this par of the theorem, we need to show that:
$\sqrt[n]{f(x)} - \sqrt[n]{A} = \gamma(x) \quad ??????$

Where $\gamma(x)$ is an arbitrary infinitesimal.

Now, lets approach this problem this way:

$$\sqrt[n]{f(x)} = [f(x)]^{\frac{1}{n}} = e^{\frac{1}{n}*ln[f(x)]}.$$

Also:

$\sqrt[n]{A} = A^{\frac{1}{n}} = e^{\frac{1}{n}ln(A)}$. This way now we have:

$$\sqrt[n]{f(x)} - \sqrt[n]{A} = e^{\frac{1}{n}*ln[f(x)]} - e^{\frac{1}{n}ln(A)} = e^{\frac{1}{n}}[f(x) - A] = \sqrt[n]{e}\,\alpha(x) =^{1)} \gamma(x).$$

1) From corollary 3.

This way part 6 of the theorem is proven.

I can also prove this way, using infinitesimals, all other theorems that have been previously proven using the Cauchy's or Heine's definition of the limit of a function.

For example let's prove one more important thing.

Let's prove that if: $\lim_{x \to a} f(x) = A$ then also $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{A}$.???

Proof:

From the definition of the limit, using infinitesimals, we have:

$f(x) - A = \alpha(x)$, where $\alpha(x)$ is an infinitesimal.

To prove this we need to show that:

$\frac{1}{f(x)} - \frac{1}{A} = \beta(x)$????

So, we have:

$$\frac{1}{f(x)} - \frac{1}{A} = \frac{A - f(x)}{Af(x)} = (-\frac{1}{A})\frac{\alpha(x)}{f(x)},$$

Now using the fact that $f(x) - A = \alpha(x) => f(x) = A + \alpha(x)$ so:

$$(-\frac{1}{A})\frac{\alpha}{f(x)} = (-\frac{1}{A})\frac{\alpha(x)}{A + \alpha(x)} = (-\frac{1}{A})\beta'(x) =^{1)} \beta(x)$$

$\beta(x)$ is an infinitesimal.
1) Corollary 3.

This actually means that : $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{A}$.

Now, let's try to prove The Squeeze theorem using our definition of limit in terms of infinitesimals.

Theorem: Let I be an interval containing the point a. Let f, g and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have:
$g(x) \le f(x) \le h(x)$, and also suppose that:

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then: $\lim_{x \to a} f(x) = L.$

Proof:

Since $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$ from the definition we have:

$g(x) - L = \alpha_1(x), \quad also \quad h(x) - L = \alpha_2(x).$

Now, from the conditions of the theorem we also have:

$g(x) \leq f(x) \leq h(x)$. Let's subtract L from this expression, so we get:

$g(x) - L \leq f(x) - L \leq h(x) - L \quad => \quad \alpha_1(x) \leq f(x) - L \leq \alpha_2(x).$

From here we can reason in two ways:

1. First, since $f(x) - L$ is between two infinitesimals, it must also be itself equal to an infinitesimal. That is,

$\alpha_1(x) \leq f(x) - L \leq \alpha_2(x) \quad => \quad f(x) - L = \alpha(x), \ \ where \ \alpha(x) \ is \ an \ infinitesimal.$

2. We may use the definition of infinitesimals to show our point. Let's go back to this expression again for a few moments:

$\alpha_1(x) \leq f(x) - L \leq \alpha_2(x) - - - - - - - - - (*)$,

since $\alpha_2(x)$ is an infinitesimal, from the definition we have:

$\forall \epsilon > 0, \exists \delta(\epsilon) > 0, \ \ such \ that \ |\alpha_2(x)| < \epsilon, \ \ whenever \ \ 0 < |x - a| < \delta.$

i) Let's further suppose that $\alpha_1(x) \geq 0$.

This way from (*) we have

$|f(x) - L| \leq |\alpha_2(x)| < \epsilon, \ => |f(x) - L| < \epsilon \ => f(x) - L = \gamma(x)$. So it yields that f(x)-L is also an infinitesimal.

ii)Now, let's suppose that $\alpha_2(x) < 0$. So this way now we have:

$\forall \epsilon > 0, \exists \delta(\epsilon) > 0, \ \ such \ that \ |\alpha_1(x)| < \epsilon, \ \ whenever \ \ 0 < |x - a| < \delta.$. Now from (*) we have:

$|f(x) - L| \leq |\alpha_1(x)| < \epsilon \quad => f(x) - L = \beta(x)$ so f(x)-L is also an infinitesimal, what proves our point.

iii) Now the other two possibilities fall in one of the above categories that we already proved. That is

$f(x) - L < 0, \ \ \alpha_2(x) > 0$ and $\alpha_1(x) < 0, \ \ f(x) - L > 0$.

This way we have proved the Squeeze Theorem using the definition of the limit of a function in terms of infinitesimals.

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