The associative property of convolution is proved in literature for infinite interval. Given below it that proof, where $\psi(t)$, $y(t)$ and $v(t)$ are valid from time $-\infty$ to ∞.

$$\psi(t) * [y(t) * v(t)] \tag{1}$$

$$\int_{-\infty}^{\infty} \psi(t_o) \left[\int_{-\infty}^{\infty} y(\tau) v(t_f - t_o - \tau) d\tau \right] dt_o \tag{2}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t_o) y(\tau) v(t_f - t_o - \tau) d\tau dt_o \tag{3}$$

Let $\lambda = \tau + t_o$. $d\lambda = dt_o$. When $t_o = -\infty$, $\lambda = -\infty$. When $t_o = \infty$, $\lambda = \infty$. Substitute $t_o = \lambda - \tau$ in (3) and we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\lambda - \tau) y(\tau) v(t_f - \lambda) d\tau d\lambda \tag{4}$$

Let $\eta = \lambda - \tau$. $d\eta = -d\tau$. When $\tau = -\infty$, $\eta = \infty$. When $\tau = \infty$, $\eta = -\infty$. Substitute $\tau = \lambda - \eta$ in (4) and we get

$$-\int_{-\infty}^{\infty} \int_{\infty}^{-\infty} \psi(\eta) y(\lambda - \eta) v(t_f - \lambda) d\eta d\lambda \tag{5}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\eta) y(\lambda - \eta) v(t_f - \lambda) d\eta d\lambda \tag{6}$$

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \psi(\eta) y(\lambda - \eta) d\eta \right] v(t_f - \lambda) d\lambda \tag{7}$$

$$[\psi(t) * y(t)] * v(t) \tag{8}$$

This is how it is done in literature.

I want to prove the associative property of convolution where $\psi(t)$, $y(t)$ and $v(t)$ are valid for finite interval, that is from time 0 to t_f.

$$\int_0^{t_f} \psi(t_o) \left[\int_0^{t_f - t_o} y(\tau) v(t_f - t_o - \tau) d\tau \right] dt_o \tag{9}$$

$$\int_0^{t_f} \int_0^{t_f - t_o} \psi(t_o) y(\tau) v(t_f - t_o - \tau) d\tau dt_o \tag{10}$$

Here i am following the same steps as before. Let $\lambda = \tau + t_o$. $d\lambda = dt_o$. When $t_o = 0$, $\lambda = \tau$. When $t_o = t_f$, $\lambda = \tau + t_f$. Substitute $t_o = \lambda - \tau$ in (10) and we get

$$\int_\tau^{t_f + \tau} \int_0^{t_f - \lambda + \tau} \psi(\lambda - \tau) y(\tau) v(t_f - \lambda) d\tau d\lambda \tag{11}$$

Let $\eta = \lambda - \tau$. $d\eta = -d\tau$. When $\tau = 0$, $\eta = \lambda$.

When $\tau = t_f - \lambda + \tau$. I get stuck here because how can this be possible.

I have tried many other ways but can not get what i want. In the end i want to see something like below which is similar to (7). The only difference is that in the equation below, the interval is finite.

$$\int_0^{t_f} \left[\int_0^{\lambda} \psi(\eta) y(\lambda - \eta) d\eta \right] v(t_f - \lambda) d\lambda \tag{12}$$

Example

I have tried one example where $\psi(t) = e^{-t}$, $y = t$ and $v = 2 - t$ which i assume is valid between 0 to t_f

$$\psi * y = \int_0^{t_f} \psi(\tau) y(t_f - \tau) d\tau = \int_0^{t_f} e^{-\tau}(t_f - \tau) d\tau = e^{-t_f} + t_f - 1 \tag{13}$$

$$[\psi * y] * v = \int_0^{t_f} (e^{-\tau} + \tau - 1)(2 - (t_f - \tau)) d\tau = \frac{-t_f^3}{6} + 1.5 t_f^2 - 3t_f - 3e^{-t_f} + 3 \tag{14}$$

$$y * v = \int_0^{t_f} y(t_f - \tau) v(\tau) d\tau = \int_0^{t_f} (t_f - \tau)(2 - \tau) d\tau = t_f^2 - \frac{t_f^3}{6} \tag{15}$$

$$\psi * [y * v] = \int_0^{t_f} e^{-(t_f - \tau)} (\tau^2 - \frac{\tau^3}{6}) d\tau = \frac{-t_f^3}{6} + 1.5 t_f^2 - 3t_f - 3e^{-t_f} + 3 \tag{16}$$

It can be seen that (14) and (16) are same, which show that assosiative property for finite intervals do work. I am not just able to prove it in general.