I use the Heaviside-Lorentz units (i.e., rationalized Gauss units).

First of all the gauge transformation should read

$$\vec{A}' = \vec{A} - \vec{\nabla}\chi, \quad \phi' = \phi + \frac{1}{c}\partial_t \chi. \tag{1}$$

Then both four-potentials create the same em. field:

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} = \vec{B}, \quad -\vec{\nabla}\phi' - \frac{1}{c}\partial_t \vec{A}' = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} = \vec{E}. \tag{2}$$

In the following we'll switch between t and $x^0 = x_0 = ct$ as is convenient. The four vector $x = (x^\mu) = (x^0, \vec{x}) = (ct, \vec{x})$.

Writing $\vec{E}$ and $\vec{B}$ in terms of the potentials makes these fields fulfill the homogeneous Maxwell equations,

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \partial_0 \vec{B} = 0, \tag{3}$$

autmoatically. Here we write $\partial_\mu = \partial / \partial x^\mu$.

The potentials are then determined by solving for the inhomogeneous Maxwell equations

$$\vec{\nabla} \times \vec{B} - \partial_0 \vec{E} = \frac{\vec{j}}{c} \Rightarrow \Box \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \partial_0 \phi) = \frac{\vec{j}}{c}, \tag{4}$$

$$\vec{\nabla} \cdot \vec{E} = \partial_0 \vec{\nabla} \cdot \vec{A} + \Delta\phi = -\rho. \tag{5}$$

Of course, there is no unique solution due to the above mentioned gauge invariance, but that's not bug but a feature! The entire first principles of electromagnetism hinge on gauge theory. It's also mathematically unavoidable, because massless vector fields as we know them from phenomenology can only be realized as gauge fields. This is derived from representation theory of the Poincare group underlying all relativistic physics.

Thus in practice you have to fix a gauge by some auxilliary condition. The Lorenz condition

$$\partial_\mu A^\mu = \partial_0 A^0 + \vec{\nabla} \cdot \vec{A} = 0, \quad (A^\mu) = (\phi, \vec{A}) \tag{6}$$

is particularly convenient, because it's manifestly Lorentz covariant and decouples the components of the four-potential, leading immediately from (4) and (5) to convenient wave equations,

$$\Box A^\mu = \frac{1}{c} j^\mu, \quad (j^\mu) = (c\rho, \vec{j}), \quad \Box = \partial_\mu \partial^\mu = \partial_0^2 - \Delta. \tag{7}$$

In (5) we use the Lorenz condition (6) to write

$$\partial_0 \vec{\nabla} \cdot \vec{A} = -\partial_0^2 A^0. \tag{8}$$

Among the solutions are particularly the retarded potentials,

$$A^\mu(x) = \frac{1}{4\pi c} \int_{\mathbb{R}^3} \frac{j^\mu(x^0 - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{9}$$

Taking the appropriate derivatives to get the field components which are unique solutions given the retardation condition you get the gauge-independent retarded electromagnetic field (known as Jefimenko equations, which are nicely discussed in the Wikipedia).

You can use other gauges. One pretty common gauge constraint is Coulomb gauge,

$$\vec{\nabla}\cdot\vec{A}_{\mathrm{C}} = 0. \tag{10}$$

This leads to the electrostatic equation for the scalar potential (which is the reason for the name "Coulomb gauge")

$$\Delta\phi_{\mathrm{C}} = -\rho, \tag{11}$$

It has a solution that looks like like in an "action-at-a-distance theory",

$$\phi_{\mathrm{C}}(t,\vec{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{12}$$

However, also the equation for the vector potential changes:

$$\Box\vec{A}_{\mathrm{C}} = \frac{1}{c}\vec{j} - \vec{\nabla}\partial_0\phi_{\mathrm{C}} = \frac{1}{c}\vec{j}_{\perp}. \tag{13}$$

We have

$$\vec{\nabla}\cdot\vec{j}_{\perp} = \vec{\nabla}\cdot\vec{j} - \partial_t\Delta\phi_{\mathrm{C}} = \vec{\nabla}\cdot\vec{j} + \partial_t\rho = 0, \tag{14}$$

due to charge conservation (continuity equation), which is an integrability condition for the Maxwell equations and must be fulfilled for consistency reasons anyway (if it's not fulfilled, also the retarded potentials in the Lorenz gauge to not give solutions of the Maxwell equations!). This shows that $\vec{j}_{\perp}$ is a solenoidal field, also called "transverse", which justifies the subscript $\perp$.

Then, if you want retarded fields, one should solve the wave equation for $\vec{A}_{\mathrm{C}}$ with the retardation condition, leading to

$$\vec{A}_{\mathrm{C}}(x) = \frac{1}{4\pi c}\int_{\mathbb{R}^3}\frac{\vec{j}_{\perp}(x^0-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{15}$$

That we get the same retarded fields becomes clear when we compare the Coulomb-gauge potentials with the Lorenz-gauge potentials.

Our goal is thus to find a scalar gauge field χ such that

$$\phi_{\mathrm{C}} = \phi + \partial_0\chi, \quad \vec{A}_{\mathrm{C}} = \vec{A} - \vec{\nabla}\chi. \tag{16}$$

To that end we use the equations of motion for the Lorenz-gauge and the Coulomb-gauge potentials (7,11,12). Acting with the box operator on the first equation of (16) leads to

$$\Box\phi_{\mathrm{C}} = (\partial_0^2 - \Delta)\phi_{\mathrm{C}} = \partial_0^2\phi_{\mathrm{C}} + \rho \stackrel{!}{=} \Box\phi + \partial_0\Box\chi = \rho + \partial_0\Box\chi \;\Rightarrow\; \partial_0\Box\chi = \partial_0^2\phi_{\mathrm{C}}. \tag{17}$$

We do not need to find a general solution for χ. Any solution consistent with the solutions for the potentials in the two gauges is sufficient. Thus we simply integrate (17) once wrt. x^0, leading to

$$\Box\chi = \partial_0\phi_{\mathrm{C}}. \tag{18}$$

Now we have solved the wave equation always in terms of the retarded Green's function. Thus we try this ansatz also for χ:

$$\chi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \frac{\partial_0 \phi_{\mathrm{C}}(x^0 - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{19}$$

This for sure fulfills the first equation (16). Nowe we have to show that it also fulfills the second equation. Thus we evaluate the gradient of (19). To that end we introduce a δ distribution to simplify the derivative, which in Eq. (19) operates both on the denominator and due to the dependence of the retarded-time argument of the integrand:

$$\vec{\nabla}\chi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^4} \mathrm{d}^4 x' \partial_0' \phi_{\mathrm{C}}(x^{0\prime}, \vec{x}') \vec{\nabla} \left(\frac{\delta(x^{0\prime} - x^0 + |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} \right). \tag{20}$$

Now the gradient operates on a function of $|\vec{x} - \vec{x}'|$ and thus we can substitute $\vec{\nabla}$ by $-\vec{\nabla}'$. Then, with an integration by parts we get

$$\vec{\nabla}\chi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^4} \mathrm{d}^4 x' \frac{\delta(x^0 - x^{0\prime} + |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|} \partial_0' \vec{\nabla}' \phi_{\mathrm{C}}(x'). \tag{21}$$

Now according to (13) $\partial_0' \vec{\nabla}' \phi_{\mathrm{C}}(x') = [\vec{j}(x') - \vec{j}_\perp(x')/c]$ and thus, after finally integrating out the δ distribution again, we get

$$\begin{aligned} \vec{\nabla}\chi(x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3 \vec{x}' \frac{\vec{j}(x^0 - |\vec{x} - \vec{x}'|, \vec{x}') - \vec{j}_\perp(x^0 - |\vec{x} - \vec{x}'|, \vec{x}')}{|\vec{x} - \vec{x}'|} \stackrel{(9,15)}{=} \vec{A}(x) - \vec{A}_{\mathrm{C}}(x) \\ &\Rightarrow \vec{A}_{\mathrm{C}}(x) = \vec{A}(x) - \vec{\nabla}\chi(x). \end{aligned} \tag{22}$$

This completes the proof that indeed χ is the gauge-transformation function between the Lorenz and Coulomb gauges. Thus also the Coulomb gauge, despite the non-retarded nature of the scalar potential ϕ_{C} in this gauge, finally leads to the same retarded solutions for the physical fields $\vec{E}$ and $\vec{B}$ as the Lorenz gauge.

This shows that the electromagnetic field is uniquely defined by Maxwell's equations and the boundary and initial conditions. To fulfill a general initial condition, of course one has to add an appropriate homogeneous solution to the linear partial differential equations of the potentials.