October 8, 2022

We consider zero free charges and currents:

$$\rho = J = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial\left(\mu \mathbf{H} + \mathbf{M}\right)}{\partial t} = i\omega\mu_{\mathbf{0}}\mathbf{H}, for\ \mathbf{M} = 0 \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial(\epsilon \mathbf{E} + \mathbf{\Delta P})}{\partial t} = -i\omega\epsilon\mathbf{E} - i\omega\Delta\mathbf{P} \tag{2}$$

The following two equations describe monochromatic waves of a lossless, unperturbed waveguide.

$$\mathbf{E}_\nu(\mathbf{r}) = \mathcal{E}_\nu(x,y)exp(i\beta_\nu z) \tag{3}$$

$$\mathbf{H}_\nu(\mathbf{r}) = \mathcal{H}_\nu(x,y)exp(i\beta_\nu z) \tag{4}$$

Generally, we know that normal modes can form a basis. So any optical field at a given frequency, can be expressed in terms of their expansion. And if we have a spatially dependent perturbation to the waveguide, we will have coupling and the amplitude will depend on z, where z is the propagation direction.

Therefore, the two equations below describe these coupled waves.

$$\mathbf{E}_\nu(\mathbf{r}) = \sum_\nu A_\nu(z)\hat{\boldsymbol{\mathcal{E}}}_{\boldsymbol{\nu}}(x,y)exp(i\beta_\nu z) \tag{5}$$

$$\mathbf{H}_\nu(\mathbf{r}) = \sum_\nu A_\nu(z)\hat{\boldsymbol{\mathcal{H}}}_{\boldsymbol{\nu}}(x,y)exp(i\beta_\nu z) \tag{6}$$

The summation is taken over all guided, radiation and evanescent modes.

We also have the "Lorentz Reciprocity Theorem"

$$\nabla \cdot \left(\mathbf{E_1} \times \mathbf{H_2^*} + \mathbf{E_2^*} \times \mathbf{H_1}\right) = -i\omega\left(\mathbf{E_1} \cdot \Delta\mathbf{P_2^*} - \mathbf{E_2^*} \cdot \Delta\mathbf{P_1}\right) \tag{7}$$

If we take E1, H1 as the fields that are described by equations 5 and 6 and E2, H2 by equations 3 and 4, we will have:

$$\mathbf{\Delta P_1} = \mathbf{\Delta P}\ and\ \mathbf{\Delta P_2} = 0 \tag{8}$$

By replacing all these into equation 7 and integrating both sides over the cross section of the waveguide, I get after some minor algebra:

$$\sum_{\nu} \nabla A_{\nu}(z) e^{i(\beta_{\nu}-\beta_{\mu})z} \left(\hat{\boldsymbol{\mathcal{E}}}_{\nu} \times \hat{\boldsymbol{\mathcal{H}}}_{\mu}^{*} + \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \times \hat{\boldsymbol{\mathcal{H}}}_{\nu} \right) = i\omega e^{-i\beta_{\mu}z} \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \Delta \boldsymbol{P}$$

$$\sum_{\nu} \nabla A_{\nu}(z) e^{i(\beta_{\nu}-\beta_{\mu})z} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \left(\hat{\boldsymbol{\mathcal{E}}}_{\nu} \times \hat{\boldsymbol{\mathcal{H}}}_{\mu}^{*} + \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \times \hat{\boldsymbol{\mathcal{H}}}_{\nu} \right) dxdy = i\omega e^{-i\beta_{\mu}z} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \Delta \boldsymbol{P} dxdy$$

And here comes the confusion. In the book, he immediately gives the result as:

$$\sum_{\nu} \frac{d}{dz} A_{\nu}(z) e^{i(\beta_{\nu}-\beta_{\mu})z} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \left(\hat{\boldsymbol{\mathcal{E}}}_{\nu} \times \hat{\boldsymbol{\mathcal{H}}}_{\mu}^{*} + \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \times \hat{\boldsymbol{\mathcal{H}}}_{\nu} \right) \cdot \hat{z} dxdy = i\omega e^{-i\beta_{\mu}z} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \hat{\boldsymbol{\mathcal{E}}}_{\mu}^{*} \Delta \boldsymbol{P} dxdy$$

How did $\hat{z}$ appear? And why we went from ∇ to $\frac{d}{dz}$? I sense it has something to do with the fact that propagation here is on the z direction. But, the transverse profiles depend on (x, y) as well. So shouldn't $\frac{d}{dx}$, $\frac{d}{dy}$matter?

Thank you in advance for your help!

Also 2 bonus questions from my long list.
They are somewhat related, but you can ignore them if you don't have the time:
1. In a anisotropic material, permittivity is a vector. But, also permittivity is defined as $\epsilon = \epsilon_r \epsilon_0$. Both $\epsilon_r \epsilon_0$ are scalars. So in the case of anisotropic material, is it actually something like this: $\Re(\epsilon) = \epsilon_r \epsilon_0$?
2. If we have a spatially dependent perturbation of the waveguide, why the amplitude does not depend on all coordinates as well? Why we are assuming that the field profile in the x, y remain the same? Shouldn't the (x, y) profile of the field in z=z0 be different with regards to a field in z=z1? Why only the amplitude changes?