1. Use Cartesian vectors in two-space to prove that the line segments joining midpoints of the consecutive sides of a quadrilateral form a parallelogram.

Let $\vec{a}A = [a_1, a_2]$

Let $\vec{b}B = [b_1, b_2]$

Let $\vec{c}C = [c_1, c_2]$

Let $\vec{d}D = [d_1, d_2]$

Midpoint of AB = E:

E :

$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

$(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2})$

Midpoint of BC: F

$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

$(\frac{b_1 + c_1}{2}, \frac{b_2 + c_2}{2})$

Midpoint of CD : G

$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

$(\frac{c_1 + d_1}{2}, \frac{c_2 + d_2}{2})$

Midpoint of DA : H

$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

$(\frac{d_1 + a_1}{2}, \frac{d_2 + a_2}{2})$

We must now find the slopes of segments EF, FG,FH, and EH

$\overrightarrow{EF}$:

$$\left[\frac{b_1 + c_1}{2} - \frac{a_1 + b_1}{2}, \frac{b_2 + c_2}{2} - \frac{a_2 + b_2}{2}\right]$$

$$\left[\frac{b_1 + c_1 - a_1 - b_1}{2}, \frac{b_2 + c_2 - a_2 - b_2}{2}\right] \qquad \text{(simplify like terms)}$$

$$\left[\frac{c_1 - a_1}{2}, \frac{c_2 - a_2}{2}\right] \qquad \text{(divide)}$$

$$\overrightarrow{EF} = \left[\frac{c_1 - a_1}{2}, \frac{c_2 - a_2}{2}\right]$$

$\overrightarrow{FG}$:

$$\left[\frac{c_1 + d_1}{2} - \frac{b_1 + c_1}{2}, \frac{c_2 + d_2}{2} - \frac{b_2 + c_2}{2}\right]$$

$$= \left[\frac{c_1 + d_1 - b_1 - c_1}{2}, \frac{c_2 + d_2 - b_2 - c_2}{2}\right]$$

$$= \left[\frac{d_1 - b_1}{2}, \frac{d_2 - b_2}{2}\right]$$

$$\overrightarrow{FG} = \left[\frac{d_1 - b_1}{2}, \frac{d_2 - b_2}{2}\right]$$

$\overrightarrow{EH}$=

$$\left[\frac{d_1 + a_1}{2} - \frac{a_1 + b_1}{2}, \frac{d_2 + a_2}{2} - \frac{a_2 + b_2}{2}\right]$$

$$= \left[\frac{d_1 + a_1 - a_1 - b_1}{2}, \frac{d_2 + a_2 - a_2 - b_2}{2}\right]$$

$$\overrightarrow{EH} = \left[\frac{d_1 - b_1}{2}, \frac{d_2 - b_2}{2}\right]$$

$\overrightarrow{HG}$:

$$\left[\frac{c_1 + d_1}{2} - \frac{d_1 + a_1}{2}, \frac{c_2 + d_2}{2} - \frac{d_2 + a_2}{2}\right]$$

$$= \left[\frac{c_1 + d_1 - d_1 - a_1}{2}, \frac{c_2 + d_2 - d_2 - a_2}{2}\right]$$

$$\overrightarrow{HG} = \left[\frac{c_1 - a_1}{2}, \frac{c_2 - a_2}{2}\right]$$

Now we check the values of the vectors:

$$\overrightarrow{EF} = \left[\frac{c_1 - a_1}{2}, \frac{c_2 - a_2}{2}\right]$$

$$\overrightarrow{FG} = \left[\frac{d_1 - b_1}{2}, \frac{d_2 - b_2}{2}\right]$$

$$\overrightarrow{EH} = \left[\frac{d_1 - b_1}{2}, \frac{d_2 - b_2}{2}\right]$$

$$\overrightarrow{HG} = \left[\frac{c_1 - a_1}{2}, \frac{c_2 - a_2}{2}\right]$$

As you can see:

$\overrightarrow{EF} = \overrightarrow{HG}$

$\overrightarrow{EF}$is parallel to $\overrightarrow{HG}$

$\overrightarrow{FG} = \overrightarrow{EH}$

$\overrightarrow{FG}$ is parallel to $\overrightarrow{EH}$

As you can see, joining the midpoints of consecutive sides of a quadrilateral forms a parallelogram.