Interest, Annuities, Sinking Funds.

In this section, n is the number of years, and r the rate of interest expressed as a decimal.

18. Amount. A principal P placed at a rate of interest r for n years accumulates to an amount A_n, as follows:

At simple interest: $A_n = P(1 + nr)$.

At interest compounded annually:* $A_n = P(1 + r)^n$.

At interest compounded q times a year: $A_n = P\left(1 + \frac{r}{q}\right)^{nq}$.

19. Nominal and Effective Rates. The rate of interest quoted in describing a given variety of compound interest is called the *nominal rate*. The rate per year at which interest is earned during each year is called the *effective rate*. The effective rate i corresponding to the nominal rate r, compounded q times a year is:

$$i = \left(1 + \frac{r}{q}\right)^q - 1.$$

20. Present or Discounted Value of a Future Amount. The present quantity P which in n years will accumulate to the amount A_n at the rate of interest r, is:

At simple interest: $P = \dfrac{A_n}{1 + nr}$.

At interest compounded annually:† $P = \dfrac{A_n}{(1 + r)^n}$.

At interest compounded q times a year: $P = \dfrac{A_n}{\left(1 + \frac{r}{q}\right)^{nq}}$.

P is called the *present value* of A_n due in n years at rate r.

21. True Discount. The true discount is:

$$D = A_n - P.$$

22. Annuity. A fixed sum of money paid at regular intervals is called an annuity.

23. Amount of an Annuity.‡ If an annuity P is deposited at the end of each successive year (beginning one year hence), and the interest at rate r, compounded annually, is paid on the accumulated deposit at the end of each year, the total amount N accumulated at the end of n years is

$$N = P \cdot \frac{(1 + r)^n - 1}{r}.$$

N is called the *amount of an annuity* P.

* See Table XIX. † See Table XX. ‡ See Table XXI.

24. Present Value of an Annuity.* The total present amount P which will supply an annuity N at the end of each year for n years, beginning one year hence, (assuming that in successive years the amount not yet paid out earns interest at rate r, compounded annually), is:

$$P = N \cdot \frac{(1 + r)^n - 1}{r(1 + r)^n} = N \cdot \frac{1 - (1 + r)^{-n}}{r}.$$

P is called the *present value* of an annuity.

25. Amount of a Sinking Fund.‡ If a fixed investment N is made at the end of each successive year (beginning at the end of the first year), and interest paid at rate r, compounded annually, is paid on the accumulated amount of the investment at the end of each year, the total amount S accumulated at the end of n years is:

$$S = N \cdot \frac{(1 + r)^n - 1}{r}.$$

S is called the *amount of the sinking fund*.

26. Fixed Investment, or Annual Installment. The amount N that must be placed at the end of each year (beginning one year hence), with compound interest paid at rate r on the accumulated deposit, in order to accumulate a sinking fund S in n years is:

$$N = S \cdot \frac{r}{(1 + r)^n - 1}.$$

N is called a *fixed investment* or *annual installment*.

Algebraic Equations†

27. Quadratic Equations. If

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If a, b, c are real and
if $b^2 - 4ac > 0$, the roots are real and unequal.
if $b^2 - 4ac = 0$, the roots are real and equal.
if $b^2 - 4ac < 0$, the roots are imaginary.

28. Cubic Equations. The cubic equation

$$y^3 + py^2 + qy + r = 0,$$

may be reduced by the substitution

$$y = \left(x - \frac{p}{3}\right)$$

* See Table XXII. ‡ Compare with § 23. † For linear equations, see § 17.

to the normal form

$$x^3 + ax + b = 0,$$

where

$$a = \frac{1}{3}(3q - p^2), \quad b = \frac{1}{27}(2p^3 - 9pq + 27r),$$

which has the solutions $\quad x_1, \ x_2, \ x_3,$

$$x_1 = A + B, \ x_2, \ x_3 = -\frac{1}{2}(A + B) \pm \frac{i\sqrt{3}}{2}(A - B),$$

where

$$i^2 = -1, \quad A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \quad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}.$$

If p, q, r are real (and hence, if a and b are real), and

if $\dfrac{b^2}{4} + \dfrac{a^3}{27} > 0$, there are one real root and two conjugate imaginary roots,

if $\dfrac{b^2}{4} + \dfrac{a^3}{27} = 0$, there are three real roots of which at least two are equal,

if $\dfrac{b^2}{4} + \dfrac{a^3}{27} < 0$, there are three real and unequal roots.

If

$$\frac{b^2}{4} + \frac{a^3}{27} < 0,$$

the above formulas are impractical. The real roots are,

$$x_k = 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\phi}{3} + 120^\circ k\right), \quad k = 0, 1, 2,$$

where

$$\cos\phi = \mp\sqrt{\frac{b^2}{4} \div \left(-\frac{a^3}{27}\right)},$$

and where the upper sign is to be used if b is positive and the lower sign if b is negative.

If

$$\frac{b^2}{4} + \frac{a^3}{27} > 0, \text{ and } a > 0,$$

the real root is,

$$x = 2\sqrt{\frac{a}{3}}\operatorname{ctn} 2\phi,$$

where ϕ and ψ are to be computed from

$$\operatorname{ctn} 2\psi = \mp\sqrt{\frac{b^2}{4} \div \frac{a^3}{27}}, \quad \tan\phi = \sqrt[3]{\tan\psi},$$

and where the upper sign is to be used if b is positive and the lower sign if b is negative.

If

$$\frac{b^2}{4} + \frac{a^3}{27} = 0,$$

the roots are,

$$x = \mp 2\sqrt{-\frac{a}{3}}, \ \pm\sqrt{-\frac{a}{3}}, \ \pm\sqrt{-\frac{a}{3}},$$

where the upper sign is to be used if b is positive, and the lower sign if b is negative.

29. Biquadratic (Quartic) Equation. The quartic equation

$$y^4 + py^3 + qy^2 + ry + s = 0$$

may be reduced to the form

$$x^4 + ax^2 + bx + c = 0$$

by the substitution

$$y = \left(x - \frac{p}{4}\right).$$

Let l, m, and n, denote the roots of the resolvent cubic

$$t^3 + \left(\frac{a}{2}\right)t^2 + \left(\frac{a^2 - 4c}{16}\right)t - \frac{b^2}{64} = 0.$$

The required roots of the reduced quartic are,

$x_1 = \pm(-\sqrt{l} - \sqrt{m} - \sqrt{n})$; $\qquad x_2 = \pm(-\sqrt{l} + \sqrt{m} + \sqrt{n})$;

$x_3 = \pm(\sqrt{l} - \sqrt{m} + \sqrt{n})$; $\qquad x_4 = \pm(\sqrt{l} + \sqrt{m} - \sqrt{n})$;

where the upper signs are to be used if b is positive, and the lower signs if b is negative.

30. General Equations of the nth degree.

$$P = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

If $n > 4$, there is no formula which gives the roots of this general equation. The following methods may be used to advantage: