PRINCIPAL STRESSES

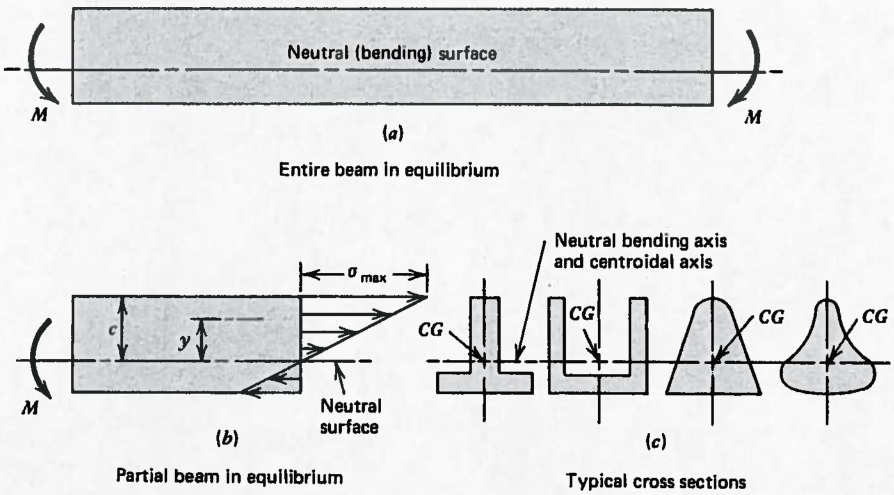

FIGURE 4.8.
Pure bending of sections with one axis of symmetry.

of symmetry. Note that the cutting-plane stresses marked "σ_{max}" are obtained from Eq. (4.6) by substituting c for y, where c is the distance from the neutral axis to the extreme fiber. Often the *section modulus* Z (defined as the ratio I/c) is used, giving the equation for maximum bending stress as

$$\sigma_{max} = M/Z \tag{4.7}$$

For a solid round bar, $I = \pi d^4/64$, $c = d/2$, and $Z = \pi d^3/32$. Hence, for this case

$$\sigma_{max} = 32M/\pi d^3 \tag{4.8}$$

Properties of various cross sections are given in Appendix B-1.

Figure 4.8 shows bending of sections having a single axis of symmetry, and where the bending moment lies in the plane containing the axis of symmetry of each cross section. At this point the reader will find it profitable to spend a few moments verifying that the offset stress distribution pattern shown is necessary to establish equilibrium in Fig. 4.8*b* (i.e., $\Sigma F = \Sigma\sigma\, dA = 0$, and $\Sigma M = M + \Sigma\sigma\, dA\, y = 0$).

4.6 PURE BENDING LOADING, CURVED BEAMS

When initially curved beams are loaded in the plane of curvature, the bending stresses are only approximately in accordance with Eqs. (4.6)–(4.8). Since the shortest (hence stiffest) path along the length of a curved beam is at the inside surface, a consideration of the relative stiffnesses of redundant load paths suggests that the stresses at the inside surface are *greater* than indicated by the straight-beam equations. Figure 4.9 illustrates that this is indeed the case. This figure also shows that equilibrium requirements result in the neutral axis shifting inward (toward the center of curvature) an amount e, and

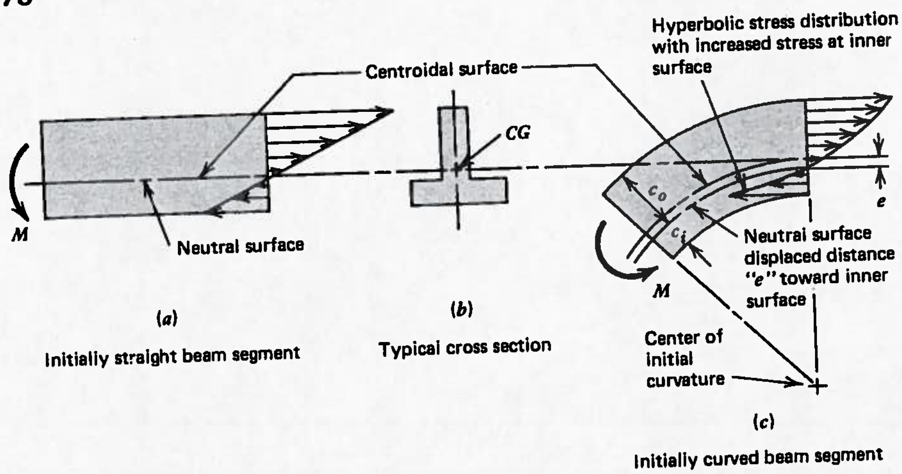

FIGURE 4.9.
Effect of initial curvature, pure bending of sections with one axis of symmetry.

the stress distribution becoming hyperbolic. These deviations from straight-beam behavior are important in severely curved beams, such as those commonly encountered in C-clamps, punch press and drill press frames, hooks, brackets, and chain links.

To understand more clearly the behavior pattern shown in Fig. 4.9*c*, let us develop the basic curved-beam stress equations. With reference to Fig. 4.10, let *abcd* represent an element bounded by plane of symmetry *ab* (which does not change direction when moment M is applied) and plane *cd*. Moment M causes plane *cd* to rotate through angle $d\phi$ to new position $c'd'$. (Note the implied assumption that initially plane sections remain plane after loading.) Rotation of this plane is, of course, about the neutral bending axis, displaced an as-yet-unknown distance e from the centroidal axis.

The strain on the fiber shown at distance y from the neutral axis is

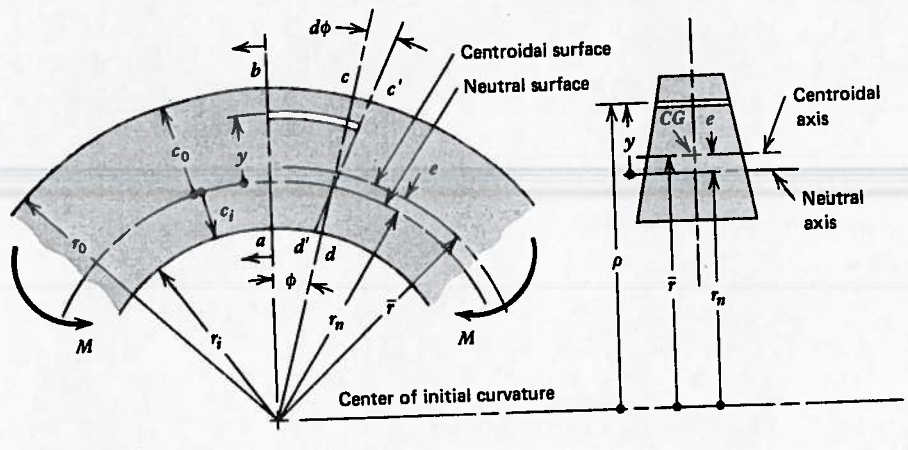

FIGURE 4.10.
Curved beam in bending.

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$$\epsilon = \frac{y\, d\phi}{(r_n + y)\phi} \tag{a}$$

For an elastic material, the corresponding stress is

$$\sigma = \frac{Ey\, d\phi}{(r_n + y)\phi} \tag{b}$$

Note that this equation gives a hyperbolic distribution of stress, as illustrated in Fig. 4.9*c*.

Equilibrium of the beam segment on either side of plane *cd* (Fig. 4.10) requires

$$\Sigma F = 0: \quad \int \sigma\, dA = \frac{E\, d\phi}{\phi} \int \frac{y\, dA}{r_n + y} = 0$$

and, since $E \neq 0$,

$$\int \frac{y\, dA}{r_n + y} = 0 \tag{c}$$

$$\Sigma M = 0: \quad \int \sigma y\, dA = \frac{E\, d\phi}{\phi} \int \frac{y^2\, dA}{r_n + y} = M \tag{d}$$

The quantity $y^2/(r_n + y)$ in Eq. (d) can be replaced by $y - r_n y/(r_n + y)$, giving

$$M = \frac{E\, d\phi}{\phi}\left[\int y\, dA - r_n \int \frac{y\, dA}{r_n + y}\right] \tag{e}$$

derivation of eq. g.

The second integral in Eq. (e) is equal to zero because of Eq. (c). The first integral is equal to *eA*. (Note that this integral would be equal to zero if *y* were measured from the centroidal axis. Since *y* is measured from an axis displaced distance *e* from the centroid, the integral has a value of *eA*.)

Substituting the above expressions into Eq. (e) gives

$$M = \frac{E\, d\phi}{\phi} eA \quad \text{or} \quad E = \frac{M\phi}{d\phi\, eA} \tag{f}$$

Substituting Eq. (f) into Eq. (b) gives

$$\sigma = \frac{My}{eA(r_n + y)} \tag{g}$$

Substituting $y = -c_i$ and $y = c_o$ in order to find maximum stress values at the inner and outer surfaces, we have

$$\sigma_i = \frac{-Mc_i}{eA(r_n - c_i)} = \frac{-Mc_i}{eAr_i}$$

$$\sigma_o = \frac{Mc_o}{eA(r_n + c_o)} = \frac{Mc_o}{eAr_o}$$

The signs of these equations are consistent with the compressive and tensile stresses produced in the inner and outer surfaces of the beam in Fig. 4.10, where the direction of moment *M* was chosen in the interest of clarifying the analysis. More commonly, a positive bending moment is defined as one tending to *straighten* an initially curved beam. In terms of this convention:

Max values

$$\underset{\text{inner}}{\sigma_i} = +\frac{Mc_i}{eAr_i} \quad \text{and} \quad \underset{\text{outer}}{\sigma_0} = -\frac{Mc_0}{eAr_0} \tag{4.9}$$

Before using Eq. (4.9), it is necessary to develop an equation for distance e. Beginning with the force equilibrium requirement, Eq. (c), and substituting ρ for $(r_n + y)$:

$$\int \frac{y\,dA}{\rho} = 0$$

But $y = \rho - r_n$; hence:

$$\int \frac{(\rho - r_n)\,dA}{\rho} = 0, \text{ or}$$

$$\int dA - \int \frac{r_n\,dA}{\rho} = 0$$

$\int dA = A$; hence,

$$A = r_n \int dA/\rho \quad \text{or} \quad r_n = \frac{A}{\int dA/\rho} \tag{h}$$

Distance e is equal to $\bar{r} - r_n$; hence:

$$e = \bar{r} - \frac{A}{\int dA/\rho} \tag{4.10}$$

Stress values given by Eq. (4.9) differ from the straight-beam "Mc/I" value by a curvature factor, K. Thus, using subscripts i and o to denote inside and outside fibers, respectively:

$$\sigma_i = K_i Mc/I = K_i M/Z \quad \text{and} \quad \sigma_0 = -K_0 Mc/I = -K_0 M/Z \tag{4.11}$$

Values of K for beams of representative cross sections and various curvatures are plotted in Fig. 4.11. This illustrates a common "rule of thumb": "If $\bar{r}$ is at least 10 times c, inner fiber stresses are usually not more than 10 percent above the Mc/I value." Values of K_0, K_i, and e are tabulated for several cross sections in [8]. Of course, any section can be handled by using Eqs. (4.9) and (4.10). If necessary, the integral in Eq. (4.10) can be evaluated graphically. Use of these equations is illustrated by the following sample problem.

SAMPLE PROBLEM 4.1

A rectangular beam has an initial curvature, $\bar{r}$, equal to the section depth, h, as shown in Fig. 4.12. How do its extreme-fiber bending stresses compare with those of an otherwise identical straight beam?

Solution

1. For the direction of loading shown in Fig. 4.12, the conventional straight-beam formula gives

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$\bar{r}$ and $\bar{c}$ explained

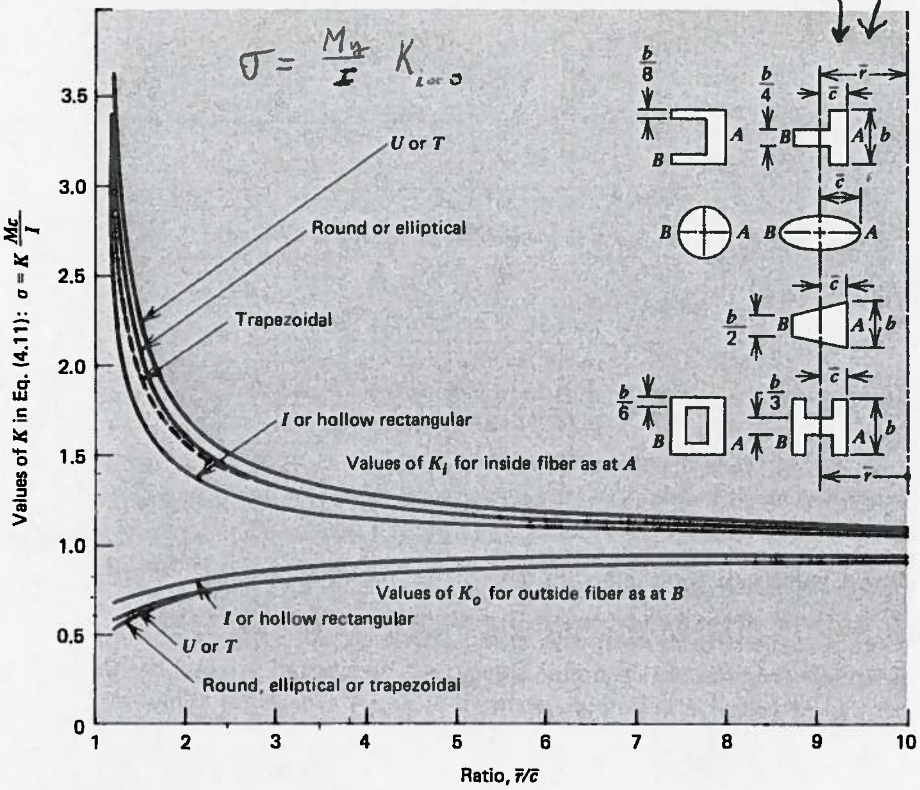

FIGURE 4.11.
Effect of curvature on bending stresses, representative cross sections. (Extracted from [8]. See section 8.1 of [8] for further details and other sections.)

$$\sigma_i = +\frac{Mc}{I} = \frac{6M}{bh^2} \qquad \sigma_0 = -\frac{6M}{bh^2}$$

2. From Eq. (4.10):

$$e = \bar{r} - \frac{A}{\int dA/\rho} = h - \frac{bh}{b\int_{r_i}^{r_0} d\rho/\rho} = h - \frac{h}{\ln(r_0/r_i)} = h\left(1 - \frac{1}{\ln 3}\right)$$

$$e = 0.089761h$$

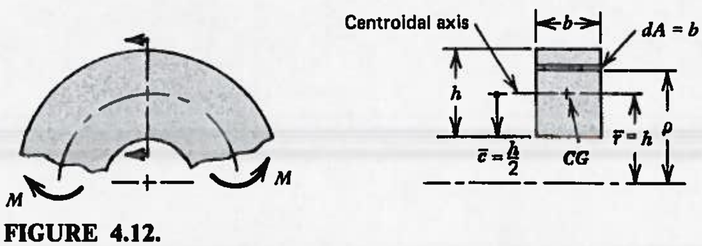

FIGURE 4.12.
Rectangular bar bent to radius of curvature, $\bar{r}$, equal to section depth, h (giving $\bar{r}/\bar{c} = 2$).

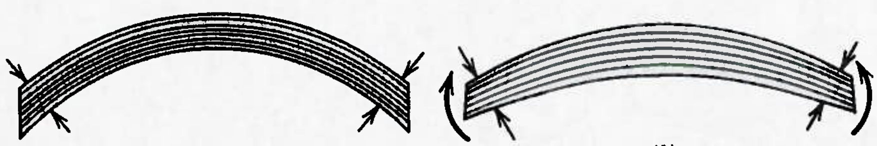

FIGURE 4.13.
Paper pad illustrating radial tension in a curved beam loaded in bending.

3. From Eq. (4.9):

$$\sigma_i = +\frac{M(0.5h - 0.089761h)}{(0.089761h)(bh)(0.5h)} = \frac{9.141M}{bh^2}$$

$$\sigma_0 = -\frac{M(0.5h + 0.089761h)}{(0.089761h)(bh)(1.5h)} = -\frac{4.380M}{bh^2}$$

4. From Eq. (4.11) with $Z = bh^2/6$:

$$K_i = \frac{9.141}{6} = 1.52 \quad \text{and} \quad K_0 = \frac{4.380}{6} = 0.73$$ ■

(These values are consistent with those shown for other sections in Fig. 4.11.)

Note that the stresses dealt with in this article are *circumferential*. Additionally, *radial* stresses are present that are, in some cases, significant. To visualize these, take a paper pad and bend it in an arc, as shown in Fig. 4.13*a*. Apply compressive forces with the thumbs and forefingers so that the sheets will not slide. Next, carefully superimpose (with the thumbs and forefingers) a small bending moment, as in 4.13*b*. Note the separation of the sheets in the center of the "beam," indicating the presence of *radial tension* (radial compression for opposite bending). These radial stresses are small if the center portion of the beam is reasonably heavy. But for an I-beam with a thin web, for example, the radial stresses can be large enough to cause damage—particularly if the beam is made of a brittle material or is subjected to fatigue loading. Further information on curved beam radial stresses is contained in [8] and [9].

4.7 TRANSVERSE SHEAR LOADING IN BEAMS

Although the *average* transverse shear stress in beams such as the shaft in Fig. 2.12 is equal to V/A (i.e., 1580 lb divided by the cross-sectional area in the critical shaft section shown in Fig. 2.12), the *maximum* shear stress is substantially higher. We will now review an analysis of the distribution of this transverse shear stress, with emphasis on an understanding of the basic concepts involved.

Figure 4.14 shows a beam of an arbitrary cross section that is symmetrical about the plane of loading. It is supported at the ends, and carries a concentrated load at the center. We wish to investigate the distribution of transverse shear stress in a plane located distance x from the left support, and at a distance y above the neutral axis. A small square element at this location is shown in the upper-right drawing. The right