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Introduction

The origin of the group determinant begins with Richard Dedekind in the late 1800's. He made several observations about the group determinant, but his success in proving these observations was limited to the case of finite Abelian groups. He shared his results with Ferdinand Georg Frobenius, who quickly became greatly interested in the topic. In fact, it was Frobenius who went on to prove many results in the case of general finite groups. The purpose of this thesis is to prove Dedekind's remarkable theorem that the group determinant of a finite Abelian group with order n is a polynomial of n variables, which can be factored into a product of linear polynomials with characters as coefficients.

In the first chapter, we will introduce some properties of characters that will be used to prove Dedekind's theorem. While the term "character" was first introduced in 1801 by Gauss in his *Disquisitiones Arithmeticae*, it wasn't until the late nineteenth century that Frobenius developed the foundations of character theory. Frobenius is credited as the creator of group character theory by Thomas Hawkins, who has studied the original correspondence that took place between Dedekind and Frobenius [5]. Perhaps most recognized as fundamental to representation theory, characters are much less known for the thing which drove Frobenius to study them in the first place- their role in the factorization of the group determinant.

The second chapter will focus on the group determinant and will contain two proofs of Dedekind's theorem, which is introduced here.

Definition. To construct the *group matrix* for an arbitrary group G, first assign an index to each element of G in any order (choosing the identity of G as the first element will be done here for convenience): $G = \{g_1, g_2, g_3, ..., g_n\}$. Create the *group matrix*, $M_G = (x_{ij})$, so that the entries are formal variables given by $\{x_{ij} = x_k : g_i g_j^{-1} = g_k\}$.

For example, take the group $\mathbb{Z}_3 = \{[0], [1], [2]\}$. Assign the g_i so that $[0] \to g_1$, $[1] \to g_2$, and $[2] \to g_3$. For the x_{ij} in the second row and first column, we have $i = 2$ and $j = 1$. Since

$g_2g_1^{-1} = g_2$, this entry is x_2. The group matrix in this case will be

$$\begin{bmatrix} x_1 & x_3 & x_2 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \end{bmatrix}.$$

Notice that the main diagonal consists of only the variable associated with the identity element. Frobenius speculated that this feature was Dedekind's motivation for using the $ij = g_ig_j^{-1}$ construction, instead of the more natural $ij = g_ig_j$ [6]. Although his results could have been obtained from either construction, the calculations are made simpler by the $ij = g_ig_j^{-1}$ construction.

Definition. The *group determinant*, $\theta(x_1, x_2, \ldots, x_n)$, is a polynomial of n independent variables.

Dedekind's theorem states that, for finite Abelian groups, θ factors into the product of linear polynomials.

Take our example $\mathbb{Z}_3$. The determinant of the group matrix given by traditional calculation in this case is

$$\begin{aligned} &x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \\ &= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3). \end{aligned} \tag{1}$$

Dedekind's factorization calls for third roots of unity, as we will see in chapter two. For the roots of unity, let $\omega = \frac{-1+i\sqrt{3}}{2}$, $\omega^2 = \frac{-1-i\sqrt{3}}{2}$, and $\omega^3 = 1$. The factorization suggested by Dedekind is

$$(x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3). \tag{2}$$

To see that this is equal to the determinant of Z_3, first, we can disregard the factor obviously shared by (1) and (2). Then,

$$
\begin{aligned}
&(x_1 + wx_2 + w^2x_3)(x_1 + w^2x_2 + wx_3) \\
&= x_1^2 + w^2x_1x_2 + wx_1x_3 + wx_1x_2 + x_2^2 + w^2x_2x_3 + w^2x_1x_3 + wx_2x_2x_3 + x_3^2 \\
&= x_1^2 + x_2^2 + x_3^2 + (x_1x_2 + x_1x_3 + x_2x_3)(w + w^2) \\
&= x_1^2 + x_2^2 + x_3^2 + (x_1x_2 + x_1x_3 + x_2x_3)(\tfrac{-1+i\sqrt{3}}{2} + \tfrac{-1-i\sqrt{3}}{2}) \\
&= x_1^2 + x_2^2 + x_3^2 + (x_1x_2 + x_1x_3 + x_2x_3)(-1) \\
&= x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3 \ .
\end{aligned}
$$

So Dedekind's theorem is verified for this example.

Chapter 1

Characters

Definition. Given a finite group G, a *character* on G is a mapping χ, from G to $\mathbb{T}$, such that $\chi(gh) = \chi(g)\chi(h)$ for all $g, h, \in G$.

Lemma 1.1. *Given a finite group G with identity e, $\chi(e) = 1$ for all characters on G.*

Proof. By the properties of characters, $\chi(e) = \chi(ee) = \chi(e)\chi(e)$. The only solution to this equation in $\mathbb{T}$ is $\chi(e) = 1$. $\square$

Lemma 1.2. *χ is a mapping to the group of n^{th} roots of unity.*

Proof. For any finite group G with order n, $g^n = e$ for all g in G. So, $\chi(g)^n = \chi(g^n) = \chi(e) = 1$. This implies that $\chi(g)$ is an n^{th} root of unity. $\square$

Lemma 1.3. *For a cyclic group G with order n, there are exactly n characters on G.*

Proof. Let G be a cyclic group generated by a, so that $G = \{a^0 = e = a^n, a^1, a^2, \ldots, a^{n-1}\}$, and let ω be a n^{th} root of unity, so that $\mu_n = \{\omega^0 = 1 = \omega^n, \omega^1, \omega^2, \ldots, \omega^{n-1}\}$. Define $\chi_k(a) := \omega^k$ for $k = 0, 1, 2, \ldots, n-1$. This gives n characters on G. To see that there are no more than n characters, suppose there is a χ_j for which $j \geq n$. Then, $j = k + cn$, for some $c \in \mathbb{N}$. Such a χ_j will map a to ω^{k+cn}, but we see that $\omega^{k+cn} = \omega^k\omega^{cn} = \omega^k$. Thus, χ_j is equivalent to χ_k. $\square$

Lemma 1.4. *The characters of a group G, with the operation multiplication defined by $\chi_i\chi_j(g) = \chi_i(g)\chi_j(g)$ for all $g \in G$, form a group, $\widehat{G}$, called the dual group of G.*

Proof. Let G be a finite group with order n, and $\widehat{G} = \{\chi_0, \chi_1, \ldots, \chi_k\}$. Because the product of n^{th} roots of unity is again an n^{th} root of unity, $\widehat{G}$ is closed. For associativity, we have

$\chi_i(\chi_j\chi_k)(g) = \chi_i(g)(\chi_j\chi_k)(g) = \chi_i(g)\chi_j(g)\chi_k(g) = (\chi_i\chi_j)(g)\chi_k(g) = (\chi_i\chi_j)\chi_k(g)$. The identity in $\widehat{G}$ is the trivial mapping $\chi_0(g) = 1$. The inverse elements $\chi(g)^{-1}$ satisfy the definition of a character since $\chi(gh)^{-1} = \chi(h^{-1}g^{-1}) = \chi(h^{-1})\chi(g^{-1}) = \chi(g)^{-1}\chi(h)^{-1}$. Thus, they are in $\widehat{G}$. $\square$

Theorem 1.5. *The number of characters for a finite Abelian group G is equal to the order of G.*

The proof of this theorem follows the reasoning given in [7]. We will use the following lemma:

Lemma 1.6. *For cyclic groups G and H, there are $|\widehat{G}| \cdot |\widehat{H}|$ distinct characters of $G \times H$.*

Proof. For cyclic groups G, with character χ_g, and H, with character χ_h, we obtain a character χ_k of $G \times H$ by $\chi_k(g,h) = \chi_g(g)\chi_h(h)$, where $g \in G$ and $h \in H$. To show that this is a homomorphism we have:

$$\begin{aligned}
\chi_k[(g_1,h_1)(g_2,h_2)] &= \chi_k(g_1g_2, h_1h_2) \\
&= \chi_g(g_1g_2)\chi_h(h_1h_2) \\
&= \chi_g(g_1)\chi_g(g_2)\chi_h(h_1)\chi_h(h_2) \\
&= \chi_g(g_1)\chi_h(h_1)\chi_g(g_2)\chi_h(h_2) = \chi_k(g_1,h_1)\chi_k(g_2,h_2).
\end{aligned}$$

To show that the characters are distinct, take characters χ_j and χ_k of $G \times H$. If $\chi_j = \chi_w\chi_x$ and $\chi_k = \chi_y\chi_z$ for non-identical pairs (χ_w, χ_x) and (χ_y, χ_z), then $\chi_j \neq \chi_k$ because either $\chi_w \neq \chi_y$, in which case there is a $g \in G$ such that $\chi_w(g) \neq \chi_y(g)$, which implies $\chi_j(g,e) \neq \chi_k(g,e)$, or, $\chi_x \neq \chi_z$, in which case there is an $h \in H$ such that $\chi_x(h) \neq \chi_z(h)$, which implies $\chi_j(e,h) \neq \chi_k(e,h)$. Thus, $G \times H$ has $|\widehat{G}| \cdot |\widehat{H}|$ characters. $\square$

Proof. (Theorem 1.5) Now, to prove that theorem 1.5 is true, it is only required to note that any finite Abelian group G can be written as the product of cyclic groups with prime power order: $G = \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_n}$, where $k_1 k_2 \cdots k_n = |G|$. Therefore, Abelian G has $|G|$ characters. $\square$

Definition. For all characters χ_i and χ_j of a dual group $\widehat{G}$, define an inner product by

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} \ . \tag{1.1}$$

Theorem 1.7. *In the vector space of all functions $f : G \to \mathbb{C}$, the characters of G are orthonormal with respect to the inner product given by* (1.1)*.*

This proof follows the presentation in [1].

Proof. Given an inner product $\langle \chi_i, \chi_j \rangle$ which satisfies the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & \text{if } \ i \neq j \\ 1 & \text{if } \ i = j \ , \end{cases}$$

the vectors χ_i and χ_j are orthonormal. To see that $\langle \chi_i, \chi_j \rangle$ satisfies δ_{ij}, first, suppose $i = j$. Then,

$$\langle \chi_i, \chi_j \rangle = \langle \chi_i, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_i(g)} = \frac{1}{|G|} \sum_{g \in G} 1 = 1 \ .$$

Next, suppose $i \neq j$. Choose an $h \in G$ such that $\chi_i(h) \neq \chi_j(h)$. Then,

$$\begin{aligned} \langle \chi_i, \chi_j \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_i(hh^{-1}g) \overline{\chi_j(hh^{-1}g)} \\ &= \chi_i(h) \overline{\chi_j(h)} \frac{1}{|G|} \sum_{g \in G} \chi_i(h^{-1}g) \overline{\chi_j(h^{-1}g)} \\ &= \chi_i(h) \overline{\chi_j(h)} \langle \chi_i, \chi_j \rangle \ . \end{aligned}$$

Because $\chi_i(h) \neq \chi_j(h)$, h cannot be equal to the identity, and $\overline{\chi_j(h)}$ cannot be the complex conjugate of $\chi_i(h)$. Thus, $\chi_i(h)\overline{\chi_j(h)}$ cannot be equal to 1. It also cannot be zero, because characters do not give zero values. Thus, $\langle \chi_i, \chi_j \rangle$ must be equal to zero. $\square$

Lemma 1.8. *Characters of G form a linearly independent vector set, $\{\chi_k(g) : \chi_k \in \hat{G}\}$.*

Proof. By the properties of the inner product (1.1), $a_i\langle \chi_i, \chi_i \rangle = 0$ implies $a_i = 0$. Let $|\widehat{G}| > n$. Assume $\sum_{i=1}^{n} a_i\chi_i = 0$, where $a_i \in \mathbb{C}$. Given an inner product of characters in $\widehat{G}$, $\langle 0, \chi_j \rangle = 0$. This implies that $\langle \sum_{i=1}^{n} a_i\chi_i, \chi_j \rangle = 0$ for all $\chi_j \in \widehat{G}$. Then, $\sum_{i=1}^{n} a_i\langle \chi_i, \chi_j \rangle = 0$. That is, $a_1\langle \chi_1, \chi_j \rangle + a_2\langle \chi_2, \chi_j \rangle + \cdots + a_n\langle \chi_n, \chi_j \rangle = 0$. By induction, when $j = n+1$, we have $\sum_{i=1}^{n} a_i\langle \chi_i, \chi_j \rangle + a_j\langle \chi_j, \chi_j \rangle = 0$. This implies, $a_j\langle \chi_j, \chi_j \rangle = 0$. Since $\langle \chi_j, \chi_j \rangle = 1$, it must be that $a_j = 0$. Thus, the characters of G are linearly independent. $\square$

Chapter 2

Factoring the Group Determinant

Finite Abelian Groups

Thereom (Dedekind). *For finite Abelian groups, θ is the product of linear polynomials,*

$$\theta = \prod_{\chi \in \hat{G}} \left[\sum_{i=1}^{n} \chi(g_i) x_i \right] . \tag{2.1}$$

Proof. For the general case of any finite Abelian group G, it must be established that $\theta \neq 0$. This is true because the term x_1^n, given by the main diagonal, will occur in θ. Since the coefficient on this term is not zero, the polynomial cannot be zero.

Next, following a proof presented by K. Conrad in [2], we will show that for each $\chi \in \widehat{G}$, $\sum_{i=1}^{n} \chi(g_i) x_i$ is a factor of θ. For a fixed χ, multiply each row by $\chi(g_i)$ and add it to the row indexed by $e = x_1$. Then, in column g_h, row e, you will have (remembering for x_{ih}, $ih = g_i g_h^{-1}$)

$$\begin{aligned}
&\chi(g_1)x_{1h} + \chi(g_2)x_{2h} + ... + \chi(g_n)x_{nh} \\
&= \sum_{i=1}^{n} \chi(g_i) x_{ih} \\
&= \chi(g_h g_h^{-1}) \sum_{i=1}^{n} \chi(g_i) x_{ih} \\
&= \chi(g_h) \sum_{i=1}^{n} \chi(g_i g_h^{-1}) x_{ih} \\
&= \chi(g_h) \sum_{i=1}^{n} \chi(g_i) x_i \ .
\end{aligned}$$

Thus, each entry in row e has a common factor of $\sum_{i=1}^{n} \chi(g_i)x_i$, so this is a factor of θ. Since there are n distinct characters in G, and characters are not scalar multiples of each other, this gives rise to n relatively prime linear polynomials in n variables. Because x_1^n is in θ, it is clear there is no coefficient missing. This completes the factorization of θ. $\square$

Definition. $\mathbb{C}[G]$ is the *group algebra* of G over the complex field and consists of all elements of the form $a_1g_1 + a_2g_2 + ... + a_ng_n$ where $g_i \in G$, and each $a_i \in \mathbb{C}$ is the coefficient corresponding to g_i. Any such element can be denoted $\sum_{i=1}^{n} a_ig_i$.

A second proof presented by Conrad compares a matrix representation of a linear map in two bases for $\mathbb{C}[G]$.

Proof. The first basis for $\mathbb{C}[G]$ is $G = \{g_1, g_2, ..., g_n\}$. The second basis is $\{\sum_{i=1}^{n} \chi(g_i)g_i\}$, $\chi \in \hat{G}$, where χ is a character and $\hat{G}$ is the dual group of G. This forms a basis for $\mathbb{C}[G]$ because the characters of G are linearly independent.

Consider the matrix of the linear map M multiplied by $\sum a_ig_i$ in the basis $\{g_1, g_2, ..., g_n\}$, defining $a_{ij^{-1}} := a_k$ such that $g_ig_j^{-1} = g_k$:

$$Mg_j := \left(\sum a_ig_i\right) g_j = \left[\sum a_{ij^{-1}}(g_ig_j^{-1})\right] g_j = \sum a_{ij^{-1}}g_i(g_j^{-1}g_j) = \sum a_{ij^{-1}}g_i \ .$$

So the matrix of M in this basis is $(a_{ij^{-1}})$.

Next, consider the same linear mapping in the basis $\{\sum \chi(g_i)g_i\}$:

$$\begin{aligned}
\left(\sum_{i=1}^{n} a_i g_i\right)\left(\sum_{j=1}^{n} \chi(g_j)g_j\right) &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_i\chi(g_j)g_ig_j\right) \\
&= \sum_{k=1}^{n}\left(\sum_{g_ig_j=g_k} a_i\chi(g_j)g_k\right) \\
&= \sum_{k=1}^{n}\left(\sum_{g_ig_j=g_k} a_i\chi(g_i^{-1}g_ig_j)\right) g_k \\
&= \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_i\chi(g_i^{-1})\chi(g_k)\right) g_k \\
&= \left(\sum_{i=1}^{n} a_i\chi(g_i^{-1})\right)\left(\sum_{k=1}^{n} \chi(g_k)g_k\right).
\end{aligned}$$

In summary,

$$\left(\sum_{i=1}^{n} a_i g_i\right)\left(\sum_{j=1}^{n} \chi(g_j)g_j\right) = \left(\sum_{i=1}^{n} a_i\chi(g_i^{-1})\right)\left(\sum_{k=1}^{n} \chi(g_k)g_k\right). \tag{2.2}$$

Recall the equation

$$MX = \lambda X \tag{2.3}$$

giving eigenvalues, λ, and associated eigenvectors, X, for a linear map, M. Comparing (2.2) to (2.3), we see that multiplication on $\sum \chi(g_j)g_j$ by $\sum a_ig_i$ gives the associated eigenvalue $\lambda = \sum a_i\chi(g_i^{-1})$. In other words, the second basis we chose for $\mathbb{C}[G]$ consists of eigenvectors for left multiplication by $\sum a_ig_i$. Again, because G is Abelian, there are n characters in $\hat{G}$, giving n distinct eigenvectors and n distinct eigenvalues. Since the determinant of a matrix is equal to the product of its eigenvalues and an $n \times n$ matrix can have at most n eigenvalues, we have

$$\det(M) = \prod_{\chi\in\hat{G}}\left(\sum_{i=1}^{n} a_i\chi(g_i^{-1})\right) = \prod_{\chi\in\hat{G}}\left(\sum_{i=1}^{n} a_i\chi(g_i)^{-1}\right) = \prod_{\chi\in\hat{G}}\left(\sum_{i=1}^{n} a_i\chi(g_i)\right).$$

Thus,

$$\det(M) = \det(a_{ij^{-1}}) = \prod_{\chi \in \hat{G}} \left(\sum_{i=1}^{n} a_i \chi(g_i) \right).$$

This is true for any a_i, therefore we can replace the a_i by formal variables x_i. That is, the polynomial $p(x_1, x_2, \ldots, x_n)$, evaluated for $x_i = a_i$, is exactly $p(a_1, a_2, \ldots, a_n)$. This implies that $\det(a_{ij^{-1}}) = \det(x_{ij^{-1}})$, or rather, in the notation of the group determinant, $\det(x_{ij})$. Because the polynomials are equivalent, we have the conclusion

$$\theta(x_1, x_2, \ldots, x_n) = \prod_{\chi \in \hat{G}} \left(\sum_{i=1}^{n} \chi(g_i) x_i \right).$$

$\square$

To demonstrate this for $\mathbb{Z}_3$, we start with the bases $\mathbb{Z}_3 = \{g_1, g_2, g_3\}$ as previously defined, and

$$\begin{aligned}
&\left\{ \sum \chi(g_i) g_i \right\} \\
&= \{\chi_1(g_1)g_1 + \chi_1(g_2)g_2 + \chi_1(g_3)g_3, \chi_2(g_1)g_1 + \chi_2(g_2)g_2 + \chi_2(g_3)g_3, \\
&\quad \chi_3(g_1)g_1 + \chi_3(g_2)g_2 + \chi_3(g_3)g_3\} \\
&= \{g_1 + g_2 + g_3, g_1 + \omega g_2 + \omega^2 g_3, g_1 + \omega^2 g_2 + \omega g_3\}.
\end{aligned}$$

Now, multiply the basis $\mathbb{Z}_3$ by $\sum a_i g_i$:

$$\begin{aligned}
&\left(\sum_{i=1}^{n} a_i g_i \right) \{g_1, g_2, g_3\} \\
&= \{a_1g_1g_1 + a_2g_2g_1 + a_3g_3g_1, a_1g_1g_2 + a_2g_2g_2 + a_3g_3g_2, a_1g_1g_3 + a_2g_2g_3 + a_3g_3g_3\} \\
&= \{a_1g_1 + a_2g_2 + a_3g_3, a_3g_1 + a_1g_2 + a_2g_3, a_2g_1 + a_3g_2 + a_1g_3\}.
\end{aligned}$$

So the matrix of M in the basis $\{g_1, g_2, g_3\}$ is given by

$$(a_{ij^{-1}}) = \begin{bmatrix} a_1 & a_3 & a_2 \\ a_2 & a_1 & a_3 \\ a_3 & a_2 & a_1 \end{bmatrix},$$

which we know from previous calculation has determinant

$$(a_1 + a_2 + a_3)(a_1 + \omega a_2 + \omega^2 a_3)(a_1 + \omega^2 a_2 + \omega a_3).$$

Applying the multiplication to the second base, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{n} a_i g_i\right) \{g_1 + g_2 + g_3, g_1 + \omega g_2 + \omega^2 g_3, g_1 + \omega^2 g_2 + \omega g_3\} \\
&= (a_1 g_1 + a_2 g_2 + a_3 g_3)\{g_1 + g_2 + g_3, g_1 + \omega g_2 + \omega^2 g_3, g_1 + \omega^2 g_2 + \omega g_3\} \\
&= \{(a_1 + a_2 + a_3)(g_1 + g_2 + g_3), (a_1 + \omega^2 a_2 + \omega a_3)(g_1 + \omega g_2 + \omega^2 g_3), \\
&\quad (a_1 + \omega a_2 + \omega^2 a_3)(g_1 + \omega^2 g_2 + \omega g_3)\}.
\end{aligned}$$

This gives elements of the form $\left(\sum_{i=1}^{n} a_i \chi(g_i^{-1})\right)\left(\sum_{i=1}^{n} \chi(h_i) h_i\right)$. Satisfying equation (2.3), the determinant of M in this basis is the product:

$$\prod_{\chi \in \hat{G}} \left(\sum a_i \chi(g_i^{-1})\right) = (a_1 + a_2 + a_3)(a_1 + \omega^2 a_2 + \omega a_3)(a_1 + \omega a_2 + \omega^2 a_3).$$

This is clearly equal to $|a_{ij^{-1}}|$, and thus $|x_{ij}|$.

General Finite Groups

Not surprisingly, perhaps, Dedekind's theorem cannot be extended to general finite groups. In the general case, irreducible non-linear factors do occur. For example, the group determinant for S_3 contains irreducible non-linear factors. This determinant was computed by Dedekind, and he sought to factor the terms further through the use of hypercomplex numbers [3], cf. [4]. To see that $\theta(S_3)$ does indeed contain irreducible non-linear factors, we will use the variable substitutions given by Dedekind in a letter to Frobenius dated April 6, 1896 [4]: $\theta(S_3) = \Phi_1\Phi_2\Phi_3^2$, where

$$\Phi_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$\Phi_2 = x_1 + x_2 + x_3 - x_4 - x_5 - x_6$$
$$\Phi_3 = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_1x_2 - x_1x_3 - x_2x_3 + x_4x_5 + x_4x_6 + x_5x_6 \ .$$

Next, using the change of variables

$$u_1 = x_1 + \omega x_2 + \omega^2 x_3, \qquad v_1 = x_4 + \omega x_5 + \omega^2 x_6,$$
$$u_2 = x_1 + \omega^2 x_2 + \omega x_3, \qquad v_2 = x_4 + \omega^2 x_5 + \omega x_6,$$

we can write Φ_3 as $(u_1u_2 - v_1v_2)$. Since this is a second degree polynomial, it can only be factored into a product of two linear polynomials. For the most general possible form, consider

$$\begin{aligned}(a + bu_1 + cu_2 + dv_1 + ev_2)&(A + Bu_1 + Cu_2 + Dv_1 + Ev_2)\\ &= C_1 + C_2u_1 + C_3u_2 + C_4v_1 + C_5v_2 + C_6u_1u_2 + C_7u_1v_1 + C_8u_1v_2\\ &\quad + C_9u_2v_1 + C_{10}u_2v_2 + C_{11}v_1v_2 + C_{12}u_1^2 + C_{13}u_2^2 + C_{14}v_1^2 + C_{15}v_2^2 \ . \qquad (2.4)\end{aligned}$$

Noting that there is no constant or linear term in Φ_3, the terms in our general factor, a and A, must be zero. Since the degree of each independent variable in Φ_3 is one, the coefficient for each variable must be zero in one of the factors. Starting with u_1, let $b = 0$. Then, since u_1u_2 does occur, it must be that $C = 0$. Next, for v_1, we can choose $d = 0$. If we chose otherwise, the proof could be continued in the same fashion. Given that v_1v_2 occurs, we must have $E = 0$. Now (2.4) becomes

$$(cu_2 + ev_2)(Bu_1 + Dv_1) = BCu_1u_2 + CDu_2v_1 + Beu_1v_2 + Dev_1v_2 \ . \tag{2.5}$$

Because u_2v_1 and u_1v_2 do not occur in Φ_3, it must be that $CDu_2v_1 = Beu_1v_2$. Since the independent variables cannot be combined, it can only be that C or D and B or e equal zero. However, no matter which two we choose, our entire polynomial (2.5) disappears. Therefore, Φ_3 is irreducible.

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