Deformations of isolated even double points of corank one (3-20-2009)

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We give a local deformation theoretic proof of Farkas' conjecture that a principally polarized complex abelian variety of dimension 4 whose theta divisor has an isolated double point of rank 3 at a point of order two is a Jacobian. The proof depends only the facts that the theta function is even, a general theta divisor is smooth, and a general singular theta divisor has only ordinary singularities. The argument yields also an explicit local normal form for the theta function near such a point.

Introduction:

Hershel Farkas conjectured [F] in 2004 the following statement designed to complete the geometric Schottky problem in genus 4: if the theta divisor on a 4 dimensional complex principally polarized abelian variety (ppav) (A, theta) has an isolated double point of rank 3 at a point of order two for the group law, then (A,theta) is a Jacobian of a smooth curve of genus 4 (which then has a vanishing even theta null). This was proved by Grushevsky and Salvati Manni in 2006 in [G-SM1] and completes the Andreotti - Mayer (and classical) program of characterizing Jacobians of genus 4 curves among all 4 dimensional ppav's by the singular points on theta.

Let A be a 4 dimensional complex ppav and "theta" a symmetric theta divisor on A, and sing(theta) its variety of singular points.

i) A is a product of lower dimensional Jacobians, iff sing(theta) has dimension 2.

ii) A is a hyperelliptic Jacobian, iff sing(theta) has dimension one.

iii) A is a non hyperelliptic Jacobian with no vanishing even theta null, iff theta has exactly 2 "conjugate" singularities, (inverses for the group law).

iv) A is a non hyperelliptic Jacobian with a vanishing even theta null, iff theta has an isolated rank 3 double point at a point of order two.

Parts i), ii), iii), are due to Beauville, and iv) to Grushevsky and Salvati Manni. See [B], Thm. p.149, (6.6) p.181, (7.4) p.184, (7.5) p.191; and [G-SM1].

We prove a local structure theorem for theta functions which implies the following statement in genus 4:

Proposition: Locally near an isolated rank 3 double point (x;s) = (0;0) of the fiber Theta(x;0) = 0 over s = 0 in C^4 x H(4), the ideal of the universal theta function of 4 variables is generated (after equivariant change of variables) by a polynomial of form $x1^2 + x2^2 + x3^2 + x4^4 + b(s)x4^2 + c(s)$, where b,c are analytic functions on Siegel space H(4) near s = 0 such that b(0) = c(0) = 0. Moreover no component of the divisor{b(s)^2-4c(s) = 0} is contained in the divisor {c(s) = 0}.

Cor: A four dimensional ppav (A,theta) whose theta divisor has an isolated double point of rank 3 at a point of order two is a Jacobian of a smooth non hyperelliptic curve of genus 4 with a vanishing even theta null.

Proof of Cor: By the theorem, in every neighborhood of s = 0, there are points s in H(4) with b(s)^2-4c(s) = 0 and $c(s) \neq 0$, over which the theta divisor has two distinct non-zero odp's {x,-x} as singularities. Since there is only one component of

the discriminant locus N(0) in A(4) over which the general singular theta divisor has more than one singularity, namely J(4) = {Jacobians and products of Jacobians}, (A,theta) lies on J(4). But (A,theta) is neither a product of Jacobians nor a hyperelliptic Jacobian, since on those ppav's theta has no isolated singularities. The double point of order two then represents the vanishing even theta null on the corresponding smooth non hyperelliptic curve. **QED.**

The idea of the proof:

Although the argument does not appeal to general theorems of deformation theory, it is motivated by them. One knows that every family of local hypersurface singularities which specializes to a given isolated singularity, is pulled back by a classifying map from one standard model family, the versal family for that singularity. A double point of "corank one" is one of the simplest singularities, defined by a function analytically equivalent to one of form (xn^2+...+x1^2 + x0^m), for m >2, and the versal deformation of this singularity is equivalent to that of the monomial x0^m, i.e. to a generic monic polynomial of degree m, [pp 187-188, A-G-V]. If the function is also even, and the singularity is at x = 0, the model is (xn^2+...+x1^2 + x0^2k), for some k ≥ 2, and the original family is equivalent to one of this form:
(xn^2+...+x1^2 + x0^2k + ak-1(s) x0^2(k-1) +....+a1(s) x0^2 + a0(s)), where all aj(0) = 0.

Thus for a theta function with an isolated double point of corank one, the model is that of a generic even polynomial y^(2k) + ak-1(s) y^2(k-1) +....+a1(s) y^2 + a0(s), specializing to the monomial y^(2k) for s = 0. The discriminant locus of this polynomial has two components: one is the locus a0 = 0, where generically there is one odp at y = 0; the other is the pullback of the discriminant locus of the polynomial t^(k) + ak-1(s) t^(k-1) +....+a1(s) t + a0(s), under the map y--->y^2 = t, where generically there are 2 odp's which are negatives of each other.

In some sense this explains why N0 has two components and the structure of generic singularities of theta on those components. In particular, near a ppav with an isolated double point of corank one at 0 on theta, the family of all ppav's is locally the pullback of the general "even" deformation of the singularity y^2k by a classifying map whose image meets a generic point of both components of the discriminant locus. For the proof, we construct a local classifying map directly, using Weierstrass preparation to produce a polynomial generator for the ideal of the theta divisor locally near the given singularity.

Preparation Lemma: Let F(x;s) be analytic near (0;0) in C^n+1 x C^r and "even in x", i.e. F(-x;s) = F(x,s), and let F(x;0) have a "rank n (or corank one) double point" at x = 0, (where if n = 0, this means F(x0;0) has no terms of degree two or less in x0).

Then there is an analytic coordinate system (z;s) = (z(x;s);s) near (0;0), which is equivariant for the minus map in x, i.e. z(-x,s) = -z(x;s), such that in these coordinates F has the form:

F(z;s) = F(x(z;s);s) = unit.(zn^2 +...+z1^2 + g(z0;s)), where the unit is analytic in (z;s) and even in z = (z0,...,zn), and g is analytic in (z0;s) and even in z0. If moreover the point x = 0 is an isolated singularity of F(x;0), then the function g may be taken to be a

polynomial in z0: g(z0;s) = z0^2k + $\sum_{0\le j\le k-1}$ aj(s) z0^2j, for some finite k ≥ 2. with analytic coefficients a(s), such that all aj(0) = 0.

Proof: After a linear change of the coordinates x in C^n+1, which we continue to denote by x, we may arrange the homogeneous quadratic term of F(x;0) to be xn^2 + a rank n-1 quadratic in (x0,...,xn-1). Then by Weierstrass preparation, there is a unique expression F(x;s) = unit.(xn^2 + b1xn + b0) where b1 and b0 are analytic on a nbhd of (0;0) in C^n x C^r and vanish at (0;0). By the uniqueness of this Weierstrass polynomial, and the evenness of F in x, it follows that b1 is odd in x, and both b0 and the unit are even in x. Now complete the square in the variable xn, by putting yn = xn + (b1)/2, so that F(x0,...,xn-1,yn;s) = unit.(yn^2 +c0) where yn is odd in x, c0(x0,...,xn-1;s) = b0 – (b1^2)/4 is even in x, and c0(x;0) has a rank n-1 double point at x = 0.

Next repeat the argument for c0 that was given for F. I.e. change (x0,...,xn-1) linearly again until the homogeneous quadratic term of c0(x;0) begins with xn-1^2, use Weierstrass, complete the square replacing xn-1 by yn-1, to get F(x0,...,xn-2,yn-1,yn;s) = unit.(yn^2 + unit(yn-1^2 +d0(x0,...,xn-2;s))) where both units and d0 are even in x, and yn-1 and yn are both odd in x. Then divide yn by an analytic, hence even, square root of the inner unit, replacing yn by zn, which is thus still odd in x.

Then F(x0,...,xn-2,yn-1,zn;s) = unit.(zn^2 + yn-1^2 + d0(x0,...,xn-2;s)), where the outer unit is the product of the two previous units, and d0 is even in x, and d0(x;0) has a rank n-2 double point at x = 0. Continuing in this way, repeatedly changing the x's linearly, applying Weierstrass, replacing the x's with y's by completing the square, and then by z's after dividing by square roots of units, we eventually come to an expression F(x0,y1,z2,...,zn;s) = unit.(zn^2 +....+ z2^2 + y1^2 + g(x0;s)), where y1 and all the zj are odd in x, and g is even in x0. If g(x0;0) is identically zero, we let z1 = y1, z0 = x0, and stop here, noting that the singular locus of F(x;0) is the smooth curve (x0,0,...,0).

If g(x0;0) is not identically zero, but vanishes at x0 = 0 to finite order 2k ≥4, then we may apply Weierstrass again to write g(x0;s) = unit.(x0^2k + $\sum_{0\le j\le (k-1)}$ aj(s) x0^2j), where the evenness of the Weierstrass polynomial in x0 and of the unit, follows from the evenness of g in x0, and the uniqueness of the Weierstrass polynomial and of the unit. Dividing y1 by a square root of the unit replaces it by z1 so that, again with a new unit out front, F(x0,z1,....,zn;s) = unit.(zn^2+...+z1^2 + x0^2k + $\sum_{0\le j<k}$ aj(s) x0^2j), for some finite k ≥ 2. Renaming x0 = z0, we have our result. **QED.**

Next we deduce a general result on theta functions of corank one. H(g) denotes Siegel space of index g, and A(g) = H(g)/Sp(g,Z) the moduli variety of g dimensional ppav's. We assume as known that the discriminant locus N(0) parametrizing ppav's in A(g) with singular theta divisor, has exactly two irreducible components for g ≥ 4, called N(0)' and Thetanull, and that a general theta divisor over N(0) has only odp's as singularities, two of them over N(0)' and one of them over Thetanull [B,De, G-SM, S-V 1,2].

Theorem: **(i)** Locally near an isolated corank one double point at (x;s) = (0;0) of the fiber Theta(x;0) = 0 over s = 0 in C^g x H(g), the ideal of the universal theta function of g variables is generated (after an equivariant change of variables) by a polynomial of form x1^2 + ... + x(g-1)^2 + xg^(2k) + ∑0≤j<k aj(s) xg^(2j), where k ≥ 2, and the aj(s) are analytic functions on H(g) near s = 0 with aj(0) = 0.
(ii) If ∂(a) is the discriminant function of the polynomial t^k + ∑0≤j<k aj t^(j), then no component of the divisor D1* = {∂(a(s)) = 0} is contained in the divisor D0* = {a0(s) = 0}.
(iii) In particular in every neighborhood of s = 0, there are points s in H(g) with ∂(a(s)) = 0 and a0(s) ≠ 0, over which the theta divisor has at least two distinct non zero points {x,-x} as singularities. Hence s = 0 lies on both components of N(0).

Proof: Part **(i)** follows from the preparation lemma and the evenness of theta. Hence there is a nbhd of (0;0) in C^g x H(g) such that the critical locus of the restriction of theta to this nbhd is locally isomorphic to a nbhd of (x;s) = (0;0) in the critical locus of an analytic family of polynomials of form (x1^2+...+x(g-1)^2 + xg^2k + ∑0≤j<k aj(s) xg^2j), for some finite k ≥ 2. The Jacobian criterion shows that the critical locus of this family is isomorphic to that of the family of even monic polynomials xg^2k + ∑0≤j<k aj(s) xg^2j of the variable xg. In particular (x1,....,xg-1,xg) is a singular point for (x1^2+...+x(g-1)^2 + xg^2k + ∑aj(s) xg^2j) if and only if x1 = ...= xg-1 = 0, and xg is a singular point of the polynomial f(xg) = xg^2k + ∑0≤j<k aj(s) xg^2j.

Thus the critical locus we want to analyze near our isolated corank one double point, is locally the pullback by the coefficient functions aj(s), of the critical locus of the generic even polynomial y^2k + ∑0≤j<k aj y^2j, near the singular point y = 0, of y^2k, i.e. near the point (y;a0,...,ak-1) = (y;a) = (0;0) in C x C^k. Thus we recall next the structure of the critical and discriminant loci of this model, the monic even polynomials of one variable.

Let V = the space of monic even polynomials y^2k + ∑0≤j<k aj y^2j of degree 2k in the variable y, parametrized by (a0,...,ak-1) in C^k ≈ V. The discriminant locus is the subset D in V of those polynomials having a repeated root, and the critical locus is the subset in C x C^k of pairs (y;a) where y is a repeated root of the polynomial with coefficient vector a. We deduce that D has exactly two irreducible components as follows. The singular points, or repeated roots, of a given even polynomial f(y) = h(y^2) are the common zeroes of h(y^2) = 0 = d/dy h(y^2) = 2y.h'(y^2).

Thus if h(t) has degree k in t, the singular points of the even polynomial h(y^2) of degree 2k in y, consist of the point y = 0 for those h(t) with zero constant term, and the two square roots y, -y, of singular points t of the polynomial h(t). Since the discriminant locus of polynomials h(t) of degree k is known to be irreducible [S-V3] and the space of polynomials with zero constant term is parametrized by the irreducible space C^k-1, thus the discriminant locus of even polynomials h(y^2) of degree 2k has two irreducible components, D0 = those h(y^2) with zero constant term, and D1 corresponding to those h(y^2) where h(t) has a repeated root.

As remarked above, the critical locus of this model family consists of pairs (y;a) in C x C^k such that y is a singular point of the polynomial with coefficient vector a. Since over D0, we have h(y^2) where h(t) is a generic polynomial with zero constant term but has no multiple roots, so the only multiple root of h(y^2) for such h is y = 0. Hence the projection map C x C^k--->C^k restricted to the critical locus has degree one over D0. Over D1, we have h(y^2) where h(t) is a general singular polynomial with only one repeated root t which in general is non zero, so the repeated roots of h(y^2) are then precisely the two square roots of t. Thus over D1 the projection from the critical locus has degree two.

Of special interest to us is the neighborhood of the point (y;a) = (0;0) in the critical locus, i.e. of the singular point y = 0 for the monomial y^2k. Since this monomial has only one root, this is the only singular point, and since $k \geq 2$ by hypothesis, this monomial lies on both components D1 and D0 of the discriminant locus. We want to examine the singularities over the component D1 near this point.

A polynomial in D1 has form f(y) = h(y^2) where h(t) has a repeated root. Hence if h(t) has a repeated root at a non zero number t, then both square roots of t are singular points of f(y) = h(y^2) so the critical locus has at least two points over this polynomial f(y). Hence the only polynomials in D1 having only one critical point over them, are those of form h(y^2) where h(t) has a singular point at t = 0, and nowhere else. This means both the constant term and linear term of h(t) are zero, hence h(t) is divisible by t^2, so h(y^2) is divisible by y^4. In particular, if f(y) = h(y^2) on D1 has only one critical point, it occurs at y = 0, which is a point of multiplicity ≥ 4.

Since the polynomial y^2k has only one singularity, at y = 0, and the projection map from the critical locus to the parameter space V of even polynomials is proper, hence all polynomials near this one have all their singularities near y = 0. I.e. given any open disc I around y = 0, there is a nbhd of y^2k such that all polynomials in this nbhd have all their singularities in I.

Since y^2k lies on both D0 and D1 in V, and a general theta divisor over H(g) is smooth, the pullbacks D0* and D1* of D0, D1 by the map s--->a(s) are both non empty (possibly reducible), divisors through 0 in Hg. Since theta has singularities over all points of D1* and D0*, both D0* and D1* are contained in the discriminant locus N(0) of H(g). We claim that no component of D1* lies entirely in Theta null.

We see this as follows: since a component Z of D1* has the same dimension as N(0), if Z lies in Theta null it would contain a generic point of Theta null, hence the singularity of theta over a generic point of Z would be a single odp, since that is the generic singularity over Theta null. But over D1* and near s = 0, the critical locus of theta contains an isomorphic copy of the full critical locus of some polynomial f(y) = y^2k + $\sum 0 \leq j < k$ aj y^2j = h(y^2) in D1. If this were only one point, we remarked above then the corresponding critical point over D1 would be at y = 0 and of multiplicity ≥ 4 for f, since f(y) = h(y^2) and h(t) is a polynomial with singularity at t = 0. Hence the singularity of theta would be that of the polynomial (x1^2+...+x(g-1)^2 + f(xg)) where

xg^4 divides f(xg). Then the singularity at x = 0 would be a corank one double point, and not an odp, which contradicts the known structure of singularities over a general point of Thetanull. Thus every component Z of D1* is contained in the other component N(0)' of N(0), where the generic singularity on theta is two odp's. Since the point s = 0 lies on some such component Z of D1*, the point s = 0 lies on both components of N(0). **QED.**

Remark: Since the map from critical to discriminant locus in the model family of polynomials was proper, the non-zero singularities {x,-x} produced in the previous argument over a general point of D1* converge to (x;s) = (0;0), as s-->0 in H(g). Hence the point s = 0 actually lies on that component of the intersection of the two components of N(0) called Rg in [De]. Thus an isolated double point of corank one at a point of order two is a limit of the two ordinary double points on some nearby singular theta divisors.

Now we can deduce the proposition originally announced. Note that when g=4, N(0)' = J(4) = {genus 4 Jacobians and products of lower genus Jacobians}.
Proposition: Locally near an isolated rank 3 double point at the origin (x;s) = (0;0) in C^4 x H(4), the ideal of the universal theta function of 4 variables is generated by the polynomial x1^2 + x2^2 + x3^2 + x4^4 +bx4^2 + c, where b(s),c(s) are analytic functions on Siegel space H(4) near s = 0, such that b(0) = c(0) = 0 and no component of the divisor{b(s)^2-4c(s) = 0} is contained in the divisor {c(s) = 0}. Thus in every neighborhood of s = 0, there are points s in H(4) with b^2-4c = 0 and c ≠ 0, over which the theta divisor has two distinct non-zero odp's {x,-x} as singularities. In particular s = 0 lies on both components of N(0), i.e. J(4) and Thetanull.
Proof: It remains only to compute k for the polynomial (x1^2 + x2^2 + x3^2 + x4^2k) which defines the genus 4 Jacobian theta divisor near the vanishing theta null. By the local algebra definition [pp.121,242, A-G-V], the Milnor number of this isolated singularity equals 2k-1. Since for a Jacobian with vanishing even theta null, we can also compute this number globally topologically [S-V1] to be 3, it follows that when g = 4 the polynomial model for the rank 3 double point has k = 2. Thus near the point (x;s) = (0;0), which is singular on the theta divisor of a genus 4 jacobian with a vanishing even theta characteristic, in suitable coordinates the theta function is associate in the local ring, to a polynomial x1^2 + x2^2 + x3^2 + x4^4 + b(s)x4^2 + c(s), with coefficients b, c, analytic on H(4). Since b^2-4c = 0, and c = 0 define the discriminant loci D1 and D0 in the space of even polynomials of degree 4, they also define the pullbacks locally in H(4), namely D1* in J(4), and D0* in Thetanull. **QED.**

Comparisons and generalizations: The proof of Farkas' conjecture by Grushevsky and Salvati Manni in [G-SM1] is a global one, establishing a containment relation between the closure of the set of genus 4 Jacobians with vanishing even theta null and the set of 4 dimensional ppav's with a double point of rank < 4 at a point of order two, comparing the degree of these varieties in a projective embedding, and invoking Bezout's theorem to conclude equality as sets.

Since their argument treats all ranks less than 4, it yields the fact also conjectured by Farkas that the presence of a non isolated double point of rank 3 on theta, implies that (A,theta) in A(4) is a product of lower dimensional Jacobians. Using this and [CM, Thm.3] implies that on an indecomposable 4 dimensional ppav A, the only possible singularities of theta are double points; and when theta is singular, an indecomposable A fails to be a Jacobian if and only if theta has a double point of rank 4 at a point of order two, if and only if all double points of theta have rank 4 and occur at points of order two.

In higher dimensions, it is proved in [G-SM2] that if $g \geq 4$, the set of ppav's of dimension g whose theta divisor has a point of order two with positive corank, equals the component of the intersection of the two irreducible components of N(0) called Rg in [De]. Their argument uses the heat equation satisfied by the theta function to show that the equations defining the two loci are the same. In a forthcoming work we propose to strengthen our local arguments to remove the hypothesis of corank one used here, and to deduce these lower rank and higher dimensional results as well from general principles valid for all general even analytic hypersurfaces.

References:

[A-M] A. Andreotti and A. Mayer, On period relations for abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa 21 (1967), 189-238.

[A-G-V] V. Arnol'd, S. Gusein - Zade, A. Varchenko, Singularities of Differentiable Maps, vol.I, Monographs in Mathematics, Birkhauser 1985, Boston Basel Stuttgart.

[B] A. Beauville, Prym varieties and the Schottky problem, Invent. Math. 41 (1977), 149-196.

[C-M] S. Casalaina-Martin, Cubic threefolds and abelian varieties of dimension five II

[D] O. Debarre: Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes, *Ann. Sc. École Norm. Sup.* **25** (1992), 687-708.

[F] H. Farkas, Vanishing theta nulls and Jacobians, in The geometry of Riemann Surfaces and Abelian Varieties, III Iberoamerican Congress on Geometry in honor of professor Sevi'n Recillas Pishmish's 60th birthday, editors JK. Munoz Porras, S. Popescu, and R. Rodriguez, Contemporary Mathematics, vol. 397, Amer. Math. Soc., 2006.

[G-SM] S. Grushevsky and R. Salvati Manni, (1) Jacobians with a vanishing theta null in genus 4, Israel J. Math. 164 (2008), 303-315; and (2) Singularities of the theta divisor at points of order two, Int. Math. Res. Not. IMRN, 2007, no.15, Art. ID, rnm 045, 15 pages.

[G-R] R. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice – Hall, 1965.

[K-S] A. Kas and M. Schlessinger, On the versal deformation of a complex space with an isolated singularity, Math. Ann. 196 (1972), 23-29.

[Mum] D. Mumford, Algebraic Geometry I, Complex Projective Varieties, Springer Verlag, 1976, Berlin, Heidelberg, New York.

[S-V1] R. Smith and R. Varley, (1) On the geometry of N(0), Rend. Sem. Mat. Univers. Politech. Torino 42, 2(1984), 29-37. (2) Components of the locus of singular theta divisors of genus 5, Algebraic Geometry, Sitges 1983, Springer Lecture Notes 1124, 338-416. (3) Then tangent cone to the discriminant, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, Pub. for Can. Math. Soc. by A.M.S., Providence, 1986, 443-460.