Deriving the S-D Equations and W-T Identities

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Ok, I will assume that you know the path integral representation of the n-point Green functions

$$\mathrm{G}(y_1, y_2, \cdots, y_n) = \frac{\langle 0 | \mathrm{T}\left(\varphi_1(y_1)\varphi_2(y_2)\cdots\varphi_n(y_n)\right) | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\int [\mathcal{D}\varphi_i]\varphi_1 \cdots \varphi_n \, \mathrm{e}^{iS[\varphi]}}{\langle 0 | 0 \rangle}$$

This is explained in most QFT textbooks. So, I will start from the following correlation function

$$\langle X \rangle = \int [\mathcal{D}\varphi_i] \, X \, \mathrm{e}^{iS[\varphi]} \ , \qquad (1.1)$$

where

$$X = \prod_i \varphi_i(y_i) = \varphi_1(y_1)\varphi_2(y_2)\cdots\varphi_n(y_n) \ . \qquad (1.2)$$

Since the φ_i's in (1.1) are dummy integration variables, we can rewrite (1.1) as

$$\langle X \rangle = \int [\mathcal{D}\varphi'_i] \, X' \, \mathrm{e}^{iS[\varphi']} \ , \qquad (1.3)$$

with $X' = \prod_i \varphi'_i(y_i)$. Now, we consider infinitesimal changes in the integration variables ($|\varepsilon^a| \ll 1$),

$$\varphi_i(y_i) \to \varphi'_i(y_i) = \varphi_i(y_i) + \varepsilon^a \, \delta_a \varphi_i(y_i) \ . \qquad (1.4)$$

To first order in ε^a, (1.4) induces the following change in X

$$X' = X + \varepsilon^a \, \delta_a X \ . \qquad (1.5)$$

We assume that the integration measure is invariant to first order in ε

$$\det \left| \frac{[\mathcal{D}\varphi'_i]}{[\mathcal{D}\varphi_j]} \right| = 1 + \mathcal{O}(\varepsilon^2) \ . \qquad (1.6)$$

This is true when the generators of (1.4) are traceless. We also assume no anomalies (the invariance of the measure breaks down for chiral transformations in theories with chiral fermions).
Inserting the equations (1.4)-(1.6) in (1.3), we find

$$\langle X \rangle = \int [\mathcal{D}\varphi_i](X + \varepsilon^a \, \delta_a X) \, \mathrm{e}^{iS[\varphi + \varepsilon \cdot \delta\varphi]} \ . \qquad (1.7)$$

Next, we expand the action to first order

$$S[\varphi+\varepsilon\cdot\delta\varphi]=S[\varphi]+\varepsilon^a\int d^4x\frac{\delta S}{\delta\varphi_i(x)}\delta_a\varphi_i(x)\ . \qquad (1.8)$$

Using this and keeping only first order terms, (1.7) becomes

$$\langle X\rangle=\int[\mathcal{D}\varphi_i]\mathrm{e}^{iS[\varphi]}\left(X+i\varepsilon^a\int d^4x\,\delta_a\varphi_i(x)\frac{\delta S}{\delta\varphi_i(x)}X(y)+\varepsilon^a\,\delta_aX(y)\right)\ . \qquad (1.9)$$

1) The Schwiger-Dyson Equations

Treating of X as a functional of the fields, we may write $\delta_a X$ as

$$\delta_aX=\int d^4x\ \delta_a\varphi_i(x)\ \frac{\delta X}{\delta\varphi_i(x)}\ . \qquad (1.10)$$

Inserting (1.10) in (1.9) and changing the order of integrations, we get

$$\langle X\rangle=\int\mathcal{D}\varphi_i\ X\,\mathrm{e}^{iS[\varphi]}+\varepsilon^a\int d^4x\,\delta_a\varphi_i(x)\left(\int\mathcal{D}\varphi_i\left(i\frac{\delta S}{\delta\varphi_i(x)}X+\frac{\delta X}{\delta\varphi_i(x)}\right)\mathrm{e}^{iS[\varphi]}\right)$$
(1.11)

On using the definition of the correlation functions as given in (1.1) we find

$$\varepsilon^a\int d^4x\,\delta_a\varphi_i(x)\left(\left\langle\frac{\delta S}{\delta\varphi_i(x)}X\right\rangle-i\left\langle\frac{\delta X}{\delta\varphi_i(x)}\right\rangle\right)=0\ . \qquad (1.12)$$

Since $\varepsilon^a\delta_a\varphi_i(x)$ are arbitrary variations, we obtain

$$\left\langle\frac{\delta S}{\delta\varphi_i(x)}X\right\rangle=i\left\langle\frac{\delta X}{\delta\varphi_i(x)}\right\rangle\ . \qquad (1.13)$$

These are the so-called Schwinger-Dyson equations. To see what they mean, consider a theory for which

$$\frac{\delta S}{\delta\varphi_i(x)}=\Gamma^{ij}(\partial^{(x)})\varphi_j(x)\ ,$$

with $\Gamma(\partial)$ being some differential operator (Dirac's, K-G's etc.). Now, if we take $X=\varphi_k(y)$ in (1.13), we find the Green's function equation of the differential operator

$$\Gamma^{ij}(\partial^{(x)})\langle\varphi_j(x)\varphi_k(y)\rangle=i\delta^i_k\,\delta^4(x-y)\ .$$

In general, Schwiger-Dyson equation implies

$$\left\langle \frac{\delta S}{\delta \varphi_i(x)} \varphi_{i_1}(x_1)\varphi_{i_2}(x_2)\cdots\varphi_{i_n}(x_n) \right\rangle = 0, \quad \forall x \notin \{x_1, x_2, \cdots, x_n\} \tag{1.14}$$

This means that the classical equation of motion is satisfied by quantum field inside a correlation function provided that its argument x^μ differs from those of all other fields.

2) The Ward-Takahashi Identities

Now, we take the transformation (1.4) to be a symmetry of the action, i.e., we have (Noether identity)

$$\frac{\delta S}{\delta \varphi_i(x)} \delta_a \varphi_i(x) = -\partial_\mu J_a^\mu(x) \ . \tag{1.15}$$

Inserting this in (1.9), we find

$$\left\langle \delta_a X(y_i) \right\rangle - i \int d^4 x \, \partial_\mu \left(\int \mathcal{D}\varphi_i \, J_a^\mu(x) \, X(y_i) \, \mathrm{e}^{iS[\varphi]} \right) = 0 \ . \tag{1.16}$$

Hence

$$\int d^4 x \, \partial_\mu \left\langle J_a^\mu(x) \, X(y_i) \right\rangle = -i \left\langle \delta_a X(y_i) \right\rangle \ . \tag{1.17}$$

Ignoring possible singular (contact) terms, we may write (1.17) as

$$\partial_\mu \left\langle J_a^\mu(x) \prod_k \varphi_k(y_k) \right\rangle = -i \sum_i \delta^4(x - y_i) \left\langle \delta_a \varphi_i(y_i) \prod_{j \neq i} \varphi_j(y_j) \right\rangle \ . \tag{1.18}$$

If $\delta\varphi_i$ is not an exact symmetry, i.e., $\partial_\mu J_a^\mu(x) \neq 0$, then we expect the following modification to (1.18)

$$\partial_\mu \left\langle J_a^\mu(x) \prod_k \varphi_k(x_k) \right\rangle = \left\langle \partial_\mu J_a^\mu(x) \prod_k \varphi_k(x_k) \right\rangle - i \sum_k \delta^4(x - x_k) \left\langle \delta_a \varphi_k(x_k) \prod_{j \neq k} \varphi_j(x_j) \right\rangle \tag{1.19}$$

This is the most general form of the Ward-Takahashi identity.

Good luck

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