Deriving the Definition of the Hyperbolic Trigonometric Functions and Their Identities

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Abstract

This document attempts to derive and explain the Hyperbolic Trigonometric Functions from a geometric perspective. Many previous versions of this same derivation exist but this document's purpose is to step through the derivation in an understandable manner. The most common definitions of the two primary hyperbolic functions $\cosh(t)$ and $\sinh(t)$ are as follows:

$$\sinh(t) = \frac{e^{\alpha} - e^{-\alpha}}{2}$$
$$\cosh(t) = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

The geometric basis for this definition is commonly forgotten , and it is in order to clarify this definition is the purpose of this composition.

Contents

1 Looking at the Basics

We begin by reviewing in a new perspective the geometric and algebraic circular trigonometric functions and applying the same concepts in the perspective of the unit hyperbola.

1.1 The basics of the circular trigonometric functions

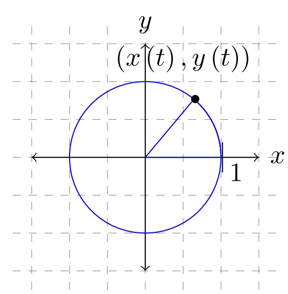
The unit circle in cartesian form is defined as:

$$x^2 + y^2 = 1 \tag{1}$$

and in parametric form is defined as:

$$x(t) = \cos t \tag{2}$$
$$y(t) = \sin t \tag{3}$$

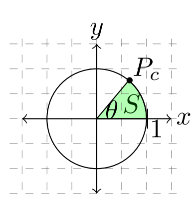
Thus, any point on the unit circle can be characterized as $(x(t), y(t))$:



For future reference, we will derive two of the three trigonometric identities (by simply substituting the equations (2) and (3) into (1) and solving for $\tan t$):

$$\cos^2 t + \sin^2 t = 1 \tag{4}$$
$$1 + \tan^2 t = \sec^2 t \tag{5}$$
$$\tan t = \sqrt{\sec^2 t - 1} \tag{6}$$

Consider a point on the unit circle P_c. The point can be described as $(x(\theta), y(\theta))$ where θ is the angle by which the segment $(1, 0)$ is swept to reach P_c, and x and y are functions defined by (2) and (3). Also consider the sector S with angle θ.



$$\begin{aligned} A_{circle} &= \pi r^2 \\ A_{unit-circle} &= \pi \\ A_{semicircle} &= \frac{\pi}{2} \end{aligned}$$

The area of the unit-circle and the area of the semicircle are specific instances of the area of sector S, A_S where $\theta = 2\pi$ and $\theta = \pi$, respectively. This is why the formula for the area of the sector formed within the unit circle is:

$$A_S = \frac{\theta}{2}$$

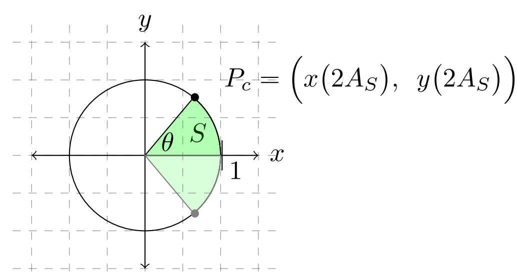
Notice that the area of the sector A_S can be calculated for this unit circle as half the angle θ. Looking at it from another perspective, the concept of the trigonometric angle is defined as twice the area that the swept segment covers.

$$\theta = 2A_S$$

Visually, the sum of the areas formed by the sector of θ and its reflection about the X axis can be used as the parameter for functions (2) and (3) (Shown below).

$$P_c = \Big(x\big(2A_S\big),\ \ y\big(2A_S\big)\Big)$$

Most importantly, we can redefine the point P_c using this property, and notice that P_c is essentially reliant on A_S rather than θ.



1.2 Looking at the Hyperbola

The unit hyperbola in cartesian form is defined as:

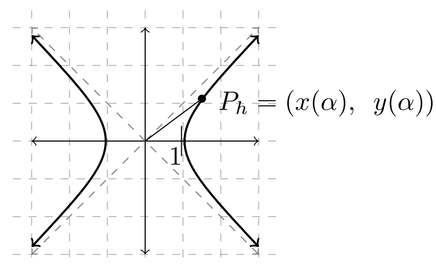
$$x^2 - y^2 = 1 \tag{7}$$

Just as the unit circle can be defined using parametric equation involving the circular trigonometric functions cos and sin, the unit hyperbola can be defined by the parametric equation involving the hyperbolic trigonometric functions cosh and sinh. This is the parametric equation for the hyperbola:

$$x(t) = \cosh t \tag{8}$$

$$y(t) = \sinh t \tag{9}$$

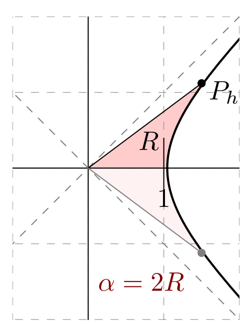
Just as the point on the unit circle P_c is defined by functions (2) and (3), the point on the unit hyperbola P_h can be described as $(x(\alpha),\ y(\alpha))$ where α is the angle created from the point $(1,0)$ to P_h and x and y are functions defined by (8) and (9).



Just like before, let us also consider the region R formed by angle α. The hyperbolic angle can be defined to be equivalent to twice the area of the region R.

$$\begin{aligned} A_R &= \frac{\alpha}{2} \\ \alpha &= 2A_R \end{aligned}$$

Visually, the reflection of region R about the x axis can be used to represent the angle α.



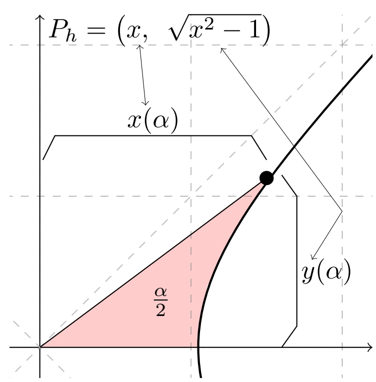
Solving for y in equation (7) we get:

$$y = \sqrt{x^2 - 1}$$

Now we can redefine the point P_h in terms of x and y:

$$P_h = \left(x,\ \sqrt{x^2 - 1}\right) = \left(x(t), y(t)\right)$$

Using this, we can find the formula for finding the value of A_R.



2 Finding and Solving the Area Formula

By finding the area of the region in terms of the x and y coordinates of the point P_h we will be able to find the equations of the horizontal and vertical components of P_h in terms of the region.

2.1 Finding the Area Formula

To find the area of the region R (A_R), we must use integration. Notice that the region R resembles the shape of a triangle, but does not include any point right of the hyperbolic function. To find the A_R we must subtract the area under the hyperbolic function (A_H) from the area of the triangle.

$$A_R = A_{triangle} - A_H$$

For

$$P_h = \left(x,\ \sqrt{x^2 - 1}\right) = \big(b, h\big)$$

The area under the hyperbola starts from where $x = 1$ to until when $x = b$, and we will integrate the function $\sqrt{x^2 - 1}$.

$$\begin{aligned} A_{triangle} &= \frac{bh}{2} \\ A_H &= \int_1^b \sqrt{x^2 - 1}\, dx \end{aligned}$$

Thus, the formula for A_R is shown to be:

$$A_R = \frac{bh}{2} - \int_1^b \sqrt{x^2 - 1}\, dx$$

2.2 Simplifying the Integral for A_H

We must solve for the variable b in A_R. Let us start by simplifying the formula for A_H.

$$\int_1^b \sqrt{x^2 - 1}\, dx$$

Looking closely, is similar to trigonometric expression in formula (6). We can attempt to simplify this integral.

$$x = \sec\theta \tag{10}$$

We'll substitute a trigonometric function of θ for x.

$$\begin{aligned}
\frac{dx}{d\theta} &= \sec\theta\tan\theta \\
dx &= \sec\theta\tan\theta\, d\theta \\
A_H &= \int_{x=1}^{x=b} \sqrt{\sec^2\theta - 1}\left(\sec\theta\tan\theta\, d\theta\right) \\
A_H &= \int_{x=1}^{x=b} \tan\theta\left(\sec\theta\tan\theta\right) d\theta
\end{aligned}$$

To exchange the lower and upper bounds of integration, we solve for θ_{lower} and θ_{upper} in equation (10).

$$\begin{aligned}
1 &= \sec\left(\theta_{lower}\right) \\
\sec^{-1}(1) &= \theta_{lower} \\
\theta_{lower} &= 0, \\
b &= \sec\left(\theta_{upper}\right) \\
\theta_{upper} &= \sec^{-1} b \\
A_H &= \int_0^{\sec^{-1} b} \tan\theta\left(\sec\theta\tan\theta\, d\theta\right)
\end{aligned}$$

The process by which we simplified is actually Trigonometric Substitution.
Using the Integration by Parts method, we can further reduce this expression. This can be to our advantage because the derivative of $\tan\theta$ is $\sec^2\theta$ and the anti-derivative of $\sec\theta\tan\theta$ is $\sec\theta$.

$$u = \tan\theta \qquad du = \sec^2\theta\, d\theta \tag{11}$$

$$dv = \sec\theta\tan\theta\, d\theta \qquad v = \sec\theta \tag{12}$$

Remembering that $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$, we will simplify A_H using equations (11) and (12).

$$\begin{aligned} A_H &= \int_0^{\sec^{-1} b} \tan^2\theta\,\sec\theta\,d\theta && (13) \\ \int_0^{\sec^{-1} b} \tan\theta\,\big(\sec\theta\tan\theta\,d\theta\big) &= \sec\theta\,\tan\theta\Bigg|_0^{\sec^{-1} b} - \int_0^{\sec^{-1} b} \sec^3\theta\,d\theta && (14) \end{aligned}$$

We will next simplify the minuend $\sec\theta\,\tan\theta\Big|_0^{\sec^{-1} b}$. Knowing that we must pass in $\sec^{-1} b$, we can manipulate the equation (6) using to make that operation simpler.

$$\sec\theta\,\tan\theta\Bigg|_0^{\sec^{-1} b} = \sec\theta\,\sqrt{\sec^2\theta - 1}\,\Bigg|_0^{\sec^{-1} b}$$

Now, we can evaluate this expression and simplify it.

$$= \sec\left(\sec^{-1} b\right)\sqrt{\sec^2\left(\sec^{-1} b\right) - 1} - \sec 0\,\sqrt{\sec^2 0 - 1}$$

Simplifying the expression becomes a simple matter of algebraic manipulation.

$$\begin{aligned} \sec 0 &= \frac{1}{\cos 0} = 1 \\ &= b\sqrt{b^2-1} - 1\sqrt{1^2-1} \\ &= b\sqrt{b^2-1} \end{aligned}$$

Now recall that the point $\big(b, h\big)$ lies on the hyperbola. Thus, it follows that h must be equivalent to $\sqrt{b^2-1}$. Thus, we can substitute bh in for $b\sqrt{b^2-1}$ and simplify equation (14).

$$A_H = bh - \int_0^{\sec^{-1} b} \sec^3\theta\,d\theta \qquad (15)$$

To simplify the subtrahend, we must recall equation (5) and substitute, and then distribute the integration.

$$\begin{aligned} \int_0^{\sec^{-1} b} \sec^3\theta\,d\theta &= \int_0^{\sec^{-1} b} \sec\theta\left(\tan^2\theta + 1\right)\,d\theta \\ &= \int_0^{\sec^{-1} b} \sec\theta\,d\theta + \int_0^{\sec^{-1} b} \sec\theta\,\tan^2\theta\,d\theta \end{aligned}$$

We must now substitute into equation (15).

$$\begin{aligned} A_H &= bh - \left(\int_0^{\sec^{-1} b} \sec\theta \, d\theta + \int_0^{\sec^{-1} b} \sec\theta \, \tan^2\theta \, d\theta \right) \\ A_H &= bh - \int_0^{\sec^{-1} b} \sec\theta \, d\theta - \int_0^{\sec^{-1} b} \sec\theta \, \tan^2\theta \, d\theta \end{aligned}$$

Since equation (13) shows A_H to be equal to the right-most subtrahend, we can create a new equation with which we can solve for the original equation.

$$\int_0^{\sec^{-1} b} \tan^2\theta \sec\theta d\theta = bh - \int_0^{\sec^{-1} b} \sec\theta \, d\theta - \int_0^{\sec^{-1} b} \sec\theta \tan^2\theta d\theta$$

Notice that we can solve for $\int_0^{\sec^{-1} b} \sec\theta \tan^2\theta \, d\theta$.

$$\begin{aligned} 2\int_0^{\sec^{-1} b} \sec\theta \tan^2\theta d\theta &= bh - \int_0^{\sec^{-1} b} \sec\theta \, d\theta \\ \int_0^{\sec^{-1} b} \sec\theta \tan^2\theta d\theta &= \frac{bh}{2} - \frac{1}{2}\int_0^{\sec^{-1} b} \sec\theta \, d\theta \end{aligned}$$

Recalling the definitions of A_R and A_H, we use them to make a new equation.

$$\begin{aligned} A_R &= \frac{bh}{2} - \int_1^b \sqrt{x^2-1} \, dx \\ \int_1^b \sqrt{x^2-1} \, dx &= \frac{bh}{2} - \frac{1}{2}\int_0^{\sec^{-1} b} \sec\theta \, d\theta \\ A_R &= \frac{bh}{2} - \left(\frac{bh}{2} - \frac{1}{2}\int_0^{\sec^{-1} b} \sec\theta \, d\theta \right) \\ A_R &= \frac{1}{2}\int_0^{\sec^{-1} b} \sec\theta \, d\theta \\ 2A_R &= \int_0^{\sec^{-1} b} \sec\theta \, d\theta \\ \alpha &= \int_0^{\sec^{-1} b} \sec\theta \, d\theta \end{aligned}$$

2.3 Simplifying the Equation for α

To further simplify this equation, we will now look at a similar problem and relate it to the original problem.

$$\begin{aligned} w &= \sec\theta + \tan\theta \\ \frac{dw}{d\theta} &= \sec\theta\,\tan\theta + \sec^2\theta \\ dw &= \sec\theta\Big(\tan\theta + \sec\theta\Big)\,d\theta \\ dw &= \sec\theta\big(w\big)\,d\theta \\ \frac{dw}{w} &= \sec\theta\,d\theta \end{aligned}$$

We can now simplify by substituting $\frac{dw}{w}$ into the original equation and solve for α.

$$\begin{aligned} \alpha &= \int_{\theta=0}^{\theta=\sec^{-1}b} \frac{dw}{w} \\ \alpha &= \ln\left(\left|w\right|\right)\Bigg|_{\theta=0}^{\theta=\sec^{-1}b} \\ \alpha &= \ln\left(\left|\sec\theta + \tan\theta\right|\right)\Bigg|_{0}^{\sec^{-1}b} \\ \alpha &= \ln\left(\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right|\right)\Bigg|_{0}^{\sec^{-1}b} \\ \alpha &= \ln\left(\left|b + \sqrt{b^2 - 1}\right|\right) - ln\left(\left|\sec 0 + \sqrt{\sec^2 0 - 1}\right|\right) \\ \alpha &= \ln\left(\left|b + \sqrt{b^2 - 1}\right|\right) - ln\left(\left|1 + \sqrt{1 - 1}\right|\right) \\ \alpha &= \ln\left(\left|b + \sqrt{b^2 - 1}\right|\right) - ln(1) \\ \alpha &= \ln\left(\left|b + \sqrt{b^2 - 1}\right|\right) \end{aligned}$$

3 Finding the Horizontal and Vertical Components of P_h

By finding the equations of b and h in terms of twice the area of the region formed by the point P_h we can find the equations to the hyperbolic functions.

3.1 Finding the Horizontal Component

Let us try solving for b by first exponentiating by e and raising by the power of two.

$$\begin{aligned}
e^{\alpha} &= \left|b + \sqrt{b^2 - 1}\right| \\
\left(e^{\alpha}\right)^2 &= \left(\left|b + \sqrt{b^2 - 1}\right|\right)^2 \\
e^{2\alpha} &= b^2 + 2b\sqrt{b^2 - 1} + b^2 - 1 \\
e^{2\alpha} &= 2b^2 + 2b\sqrt{b^2 - 1} - 1 \\
e^{2\alpha} &= 2b(b + \sqrt{b^2 - 1}) - 1 \\
e^{2\alpha} &= 2b(e^{\alpha}) - 1 \\
\frac{e^{2\alpha}}{e^{\alpha}} &= \frac{2b(e^{\alpha}) - 1}{e^{\alpha}} \\
e^{\alpha} &= 2b - e^{-\alpha}
\end{aligned}$$

Solving for b we arrive at:

$$b = \frac{e^{\alpha} + e^{-\alpha}}{2} \tag{16}$$

Recalling from equations (8) and (9), we can see that the horizontal component of any point on the unit hyperbola is defined by $\cosh t$ and by showing that b is equivalent to that horizontal component, we have shown that:

$$\cosh t = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

3.2 Finding the Vertical Component

Knowing b we can find the vertical component by recalling the equation (16) and substituting b for x.

$$\begin{aligned}
h &= \sqrt{b^2 - 1} \\
h &= \sqrt{\left(\frac{e^{\alpha} + e^{-\alpha}}{2}\right)^2 - 1} \\
h &= \sqrt{\frac{1}{4}\left[\left(e^{\alpha} + e^{-\alpha}\right)^2 - 4\right]} \\
h &= \frac{1}{2}\sqrt{\left(e^{\alpha} + e^{-\alpha}\right)^2 - 4} \\
h &= \frac{1}{2}\sqrt{\left(e^{2\alpha} + 2e^{\alpha}e^{-\alpha} + e^{-2\alpha}\right) - 4} \\
h &= \frac{1}{2}\sqrt{\left(e^{2\alpha} + 2 + e^{-2\alpha}\right) - 4} \\
h &= \frac{1}{2}\sqrt{e^{2\alpha} - 2 + e^{-2\alpha}} \\
h &= \frac{1}{2}\sqrt{\left(e^{\alpha} - e^{-\alpha}\right)^2} \\
h &= \frac{e^{\alpha} - e^{-\alpha}}{2}
\end{aligned}$$

Thus, since the vertical component h can be represented as a function of twice the area of the region swept by the point P_h, we have found the equation for $\sinh t$.

$$\sinh t = \frac{e^{\alpha} - e^{-\alpha}}{2}$$