for an r-blade A_r we have (the proof is left to the reader)

$$\boldsymbol{e}_i\boldsymbol{e}^i \cdot A_r = rA_r \tag{1.71}$$

Since $\boldsymbol{e}_i\boldsymbol{e}^i = n$ we have

$$\boldsymbol{e}_i\boldsymbol{e}^i \wedge A_r = \boldsymbol{e}_i\left(\boldsymbol{e}^i A_r - \boldsymbol{e}^i \cdot A_r\right) = (n-r)\,A_r \tag{1.72}$$

Flipping $\boldsymbol{e}^i$ and A_r in equations 1.71 and 1.72 and subtracting equation 1.71 from 1.72 gives

$$\boldsymbol{e}_i A_r \boldsymbol{e}^i = (-1)^r\,(n-2r)\,A_r \tag{1.73}$$

In Hestenes and Sobczyk (3.14) it is proved that

$$\left(\boldsymbol{e}^{k_r} \wedge \ldots \wedge \boldsymbol{e}^{k_1}\right) \cdot \left(\boldsymbol{e}_{j_1} \wedge \ldots \wedge \boldsymbol{e}_{j_r}\right) = \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\ldots\delta^{j_r}_{k_r} \tag{1.74}$$

so that the general multivector A can be expanded in terms of the blades of the frame and reciprocal frame as

$$A = \sum_{i<j<\cdots<k} A_{ij\cdots k}\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k \tag{1.75}$$

where

$$A_{ij\cdots k} = \left(\boldsymbol{e}_k \wedge \cdots \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_i\right) \cdot A \tag{1.76}$$

The components $A_{ij\cdots k}$ are totally antisymmetric on all indices and are usually referred to as the components of an *antisymmetric tensor*.

1.15 Linear Transformations

1.15.1 Definitions

Let f be a linear transformation on a vector space $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$ $\forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \Re$. Then define the action of f on a blade of the geometric algebra by

$$f(a_1 \wedge \ldots \wedge a_r) = f(a_1) \wedge \ldots \wedge f(a_1) \tag{1.77}$$

and the action of f on any two $A, B \in \mathcal{G}(\mathcal{V})$ by

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \tag{1.78}$$

Since any multivector A can be expanded as a sum of blades $f(A)$ is defined. This has many consequences. Consider the following definition for the determinant of f, $\det(f)$.

$$f(I) = \det(f)\, I \tag{1.79}$$

First show that this definition is equivalent to the standard definition of the determinant (again $e_1, \ldots, e_N$ is an orthonormal basis for $\mathcal{V}$).

$$f(e_r) = \sum_{s=1}^{N} a_{rs} e_s \tag{1.80}$$

Then

$$\begin{aligned} f(I) &= \left(\sum_{s_1=1}^{N} a_{1s_1} e_s\right) \wedge \ldots \wedge \left(\sum_{s_N=1}^{N} a_{Ns_N} e_s\right) \\ &= \sum_{s_1,\ldots,s_N} a_{1s_1} \ldots a_{Ns_N} e_{s_1} \ldots e_{s_N} \end{aligned} \tag{1.81}$$

But

$$e_{s_1} \ldots e_{s_N} = \varepsilon^{s_1 \ldots s_N}_{1 \ldots N} e_1 \ldots e_N \tag{1.82}$$

so that

$$f(I) = \sum_{s_1,\ldots,s_N} \varepsilon^{s_1 \ldots s_N}_{1 \ldots N} a_{1s_1} \ldots a_{Ns_N} I \tag{1.83}$$

or

$$\det(f) = \sum_{s_1,\ldots,s_N} \varepsilon^{s_1 \ldots s_N}_{1 \ldots N} a_{1s_1} \ldots a_{Ns_N} \tag{1.84}$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned} \det(fg)\, I &= fg(I) \\ &= f(g(I)) \\ &= f(\det(g)\, I) \\ &= \det(g)\, f(I) \\ &= \det(g) \det(f)\, I \end{aligned} \tag{1.85}$$

or

$$\det(fg) = \det(f) \det(g) \tag{1.86}$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?