see also

The Dirac Delta Function, $\delta(x-x_o)$

Dirac Delta Function

In one dimension, $\delta(x-x_o)$ is defined to be such that:

$$\int_{a \text{ to } b} f(x)\, \delta(x-x_o)dx \equiv \begin{cases} 0 & \text{if } x_o \text{ is not in } [a,b]. \\ \tfrac{1}{2}f(x_o) & \text{if } x_o = a \text{ or } b; \\ f(x_o) & \text{if } x_o\ \varepsilon\ (a,b). \end{cases}$$

Properties of $\delta(x-x_o)$: (you should know those marked with *)

*1. $\delta(x-x_o) = 0 \quad \text{if } x \neq x_o$

*2. $\int_{-\infty \text{ to } +\infty} \delta(x)dx = 1$

3. $\delta(ax) = \delta(x)/|a|$

*4. $\delta(-x) = \delta(x)$

5. $\delta(x^2-a^2) = [\delta(x-a) + \delta(x+a)]/(2a); \quad a \geq 0$

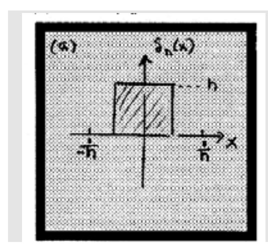
6. $\int_{-\infty \text{ to } +\infty} \delta(x-a)\delta(x-b)dx = \delta(a-b)$

*7. $\delta(g(x)) = \sum_i \delta(x-x_{oi})/|dg/dx|_{x=xoi}$ where $g(x_{oi}) = 0$ and dg/dx exists at and in a region around x_{oi}.

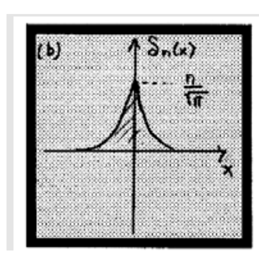
*8. $f(x)\delta(x-a) = f(a)\delta(x-a)$

9. $\delta(x)$ is a "symbolic" function which provides convenient notation for many mathematical expressions. Often one "uses" $\delta(x)$ in expressions which are not integrated over. However, it is understood that eventually these expressions will be integrated over so that the definition of δ (box above) applies.

10. No ordinary function having exactly the properties of $\delta(x)$ exists. However, one can **approximate** $\delta(x)$ by the limit of a sequence of (non-unique) functions, $\delta_n(x)$. Some examples of $\delta_n(x)$ which work are given below. In all these cases, $\int_{-\infty \text{ to } +\infty} \delta_n(x)dx = 1\ \forall\ n$ and $\text{limit}_{n \to \infty} \int_{-\infty \text{ to } +\infty} \delta_n(x-x_o)f(x)dx = f(x_o).\ \forall\ n.$



(a) $\delta_n(x) \equiv \begin{cases} 0 & \text{for } x < -1/(2n) \\ n & \text{for } -1/(2n) \leq x \leq 1/(2n) \\ 0 & \text{for } x > 1/(2n) \end{cases}$



(b) $\delta_n(x) \equiv n/\sqrt{\pi}\ \exp[-n^2x^2]$

(c) $\delta_n(x) \equiv (n/\sqrt{\pi})\cdot 1/(1 + n^2x^2)$

(d) $\delta_n(x) \equiv \sin(nx)/\pi x = [1/(2\pi)]\int_{-n \text{ to } n} \exp(ixt)dt$

11. $\int_{a\ to\ b} f(x)\, d^r/dx^r\, \delta(x-x_o)\, dx = \begin{cases} (-1)^r\, d^r f/dx^r|_{xo} & \text{if } x_o\ \varepsilon\ (a,b) \\ \tfrac{1}{2}[(-1)^r d^r f/dx^r|_{xo} & \text{if } x_o = a \text{ or } b \\ 0 & \text{otherwise} \end{cases}$

f(x) is arbitrary, continuous function at $x = x_o$

12. $\int_{a\ to\ b} x^r\, d^r/dx^r\, \delta(x-x_o)dx = \int_{a\ to\ b} (-1)^r r!\, \delta(x-x_o)\, dx$ where $x_o\ \varepsilon\ (a,b)$.

important expressions involving $\delta(x-x_o)$

13. $\delta(x-x_o) = [1/2\pi] \int_{-\infty\ to\ +\infty} e^{ik(x-xo)} dk$

14. $\delta(\mathbf{r}-\mathbf{r}_o) \equiv \delta(x-x_o)\delta(y-y_o)\delta(z-z_o) = [1/2\pi]^3 \iiint_{all\ k\ space} \exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_o)]dk_x dk_y dk_z$

15. $\delta(g(x)) = \sum_n \delta(x-\beta_n)/|dg/dx|_{x=\beta n}$ where $g(\beta_n) = 0$.

16. $\delta(\mathbf{r}-\mathbf{r}_o) = \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}$ in general system.

Dirac Delta Function in 3 Dimensions: $\delta(\mathbf{r} - \mathbf{r}_o) \equiv \delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$

17. $\delta(\mathbf{k} - \mathbf{k}_o) = [1/2\pi]^3 \int_{-\infty\ to\ +\infty}\int_{-\infty\ to\ +\infty}\int_{-\infty\ to\ +\infty} \exp[i\mathbf{r}\cdot(\mathbf{k} - \mathbf{k}_o)]\, dxdydz$

18. $\delta(\mathbf{r} - \mathbf{r}_o) = \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}$

derivation: $\iiint \delta(\mathbf{r} - \mathbf{r}_o)f(\mathbf{r})\, d^3x = f(\mathbf{r}_o) = \iiint \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}\cdot f(\mathbf{r}(q^i))\sqrt{g}dq^1dq^2dq^3$

19. $\delta(\mathbf{r} - \mathbf{r}_o)_{spherical\ coordinates} = \delta(r-r_o)\delta(\varphi-\varphi_o)\delta(\theta-\theta_o)/(r^2\sin\theta)$

20. $\tfrac{1}{2}[\delta(x-a)+\delta(x+a)] = [1/\pi] \int_{0\ to\ \infty} \cos(ka) \cos(kx)\, dk$

Exercises: Evaluate the following integrals.

a) $\int_{-1\ to\ 5} \delta(2x-\pi) \exp[\sin^3(x-\pi)]\, dx$

b) $\iiint_{all\ space} \delta(\mathbf{r}\cdot\mathbf{r}-a^2)\delta(\cos\theta-1/\sqrt{2})\delta(\sin\varphi-\tfrac{1}{2})\exp[i\mathbf{k}\cdot\mathbf{r}]d^3x$

Helmholtz Theorem

If

(a) $\nabla \cdot \mathbf{F}(\mathbf{r}) = \rho(\mathbf{r})$ everywhere for finite r;
(b) $\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r})$ "
(c) $\text{limit}_{r \to \infty} \rho(\mathbf{r}) = 0$;
(d) $\text{limit}_{r \to \infty} |\mathbf{J}(\mathbf{r})| = 0$;

then

$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r})$

where Φ and $\mathbf{A}$ are determined from ρ and $\mathbf{J}$ as shown below.

proof:

1. Define Φ and $\mathbf{A}$ as follows:

$$\Phi(\mathbf{r}) = [1/4\pi]\iiint_{V=\text{all space}} \rho(\mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\, d^3x' + \Phi_o(\mathbf{r}), \quad \text{where } \nabla^2\Phi_o(\mathbf{r}) = 0;$$

$$\mathbf{A}(\mathbf{r}) = [1/4\pi]\iiint_{V=\text{all space}} \mathbf{J}(\mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\, d^3x' + \mathbf{A}_o(\mathbf{r}), \quad \text{where } \nabla\text{x}(\nabla\text{x}\mathbf{A}_o(\mathbf{r})) = 0;$$

2. Let $\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})$; we shall show that this $\mathbf{F}$ satisfies the conditions (a) and (b) if (c) and (d) hold:

$$\nabla \cdot \mathbf{F} = \nabla \cdot [-\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})] = -\nabla^2\Phi(\mathbf{r})$$
$$\nabla \times \mathbf{F} = \nabla\text{x}[-\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})] = \nabla\text{x}[\nabla\text{x}\mathbf{A}(\mathbf{r})].$$

3. $\nabla \cdot \mathbf{F} = -\nabla^2\Phi(\mathbf{r}) = -[1/4\pi]\iiint_{V=\text{all space}} \rho(\mathbf{r'})\nabla^2[1/|\mathbf{r} - \mathbf{r'}|]\, d^3x' + 0$

4. But $\nabla^2[1/|\mathbf{r} - \mathbf{r'}|]$
$= \nabla\cdot\nabla[1/|\mathbf{r} - \mathbf{r'}|] = \nabla\cdot[-(\mathbf{r} - \mathbf{r'})/|\mathbf{r} - \mathbf{r'}|^3]$
$= -[3/|\mathbf{r} - \mathbf{r'}|^3 + [-3/|\mathbf{r} - \mathbf{r'}|^4](\mathbf{r} - \mathbf{r'})\cdot(\mathbf{r} - \mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\,]$
$= -[3/|\mathbf{r} - \mathbf{r'}|^3 + [-3/|\mathbf{r} - \mathbf{r'}|^3]\,]$
$= 0 \quad$ if $\mathbf{r} \neq \mathbf{r'}$

5 What happens if $\mathbf{r} = \mathbf{r'}$? We shall see that the expression --> ∞, but with a crucial additional property!

$$\textit{CLAIM}: \quad \nabla^2 \frac{1}{|r-r'|} = -4\pi\, \delta(r-r')$$

Derivation:
(a) Consider the following integral, $\iiint_V \nabla\cdot\nabla[1/r]\, d^3x = \iiint_V \nabla^2[1/r]\, d^3x$
where V = all space and V' = all space except a sphere of radius δ centered on the origin and a "funnel" extending from $\mathbf{r} = 0$ to $r = \infty$. See figure below.

We shall show that this integral = -4π if $\mathbf{r} = 0$ is in V.
Note: V contains $\mathbf{r} = 0$ and V' does not.