Proposition 19. Let R be a commutative ring with 1.

(1) Prime ideals are primary.

(2) The ideal Q is primary if and only if every zero divisor in R/Q is nilpotent.

(3) If Q is primary then rad Q is a prime ideal, and is the unique smallest prime ideal containing Q.

(4) If Q is an ideal whose radical is a maximal ideal, then Q is a primary ideal.

(5) Suppose M is a maximal ideal and Q is an ideal with $M^n \subseteq Q \subseteq M$ for some $n \geq 1$. Then Q is a primary ideal with rad $Q = M$.

Proof: The first two statements are immediate from the definition of a primary ideal. For (3), suppose $ab \in \text{rad } Q$. Then $a^m b^m = (ab)^m \in Q$, and since Q is primary, either $a^m \in Q$, in which case $a \in \text{rad } Q$, or $(b^m)^n \in Q$ for some positive integer n, in which case $b \in \text{rad } Q$. This proves that rad Q is a prime ideal, and it follows that rad Q is the smallest prime ideal containing Q (Proposition 12).

To prove (4) we pass to the quotient ring R/Q; by (2), it suffices to show that every zero divisor in this quotient ring is nilpotent. We are reduced to the situation where $Q = (0)$ and $M = \text{rad } Q = \text{rad}(0)$, which is the nilradical, is a maximal ideal. Since the nilradical is contained in every prime ideal (Proposition 12), it follows that M is the unique prime ideal, so also the unique maximal ideal. If d were a zero divisor, then the ideal (d) would be a proper ideal, hence contained in a maximal ideal. This implies that $d \in M$, hence every zero divisor is indeed nilpotent.

Finally, suppose $M^n \subseteq Q \subseteq M$ for some $n \geq 1$ where M is a maximal ideal. Then $Q \subseteq M$ so rad $Q \subseteq \text{rad } M = M$. Conversely, $M^n \subseteq Q$ shows that $M \subseteq \text{rad } Q$, so rad $Q = M$ is a maximal ideal, and Q is primary by (4).

Definition. If Q is a primary ideal, then the prime ideal $P = \text{rad } Q$ is called the *associated prime* to Q, and Q is said to *belong* to P (or to be P-*primary*).

It is easy to check that a finite intersection of P-primary ideals is again a P-primary ideal (cf. the exercises).

Examples

(1) The primary ideals in $\mathbb{Z}$ are 0 and the ideals (p^m) for p a prime and $m \geq 1$.

(2) For any field k, the ideal (x) in $k[x, y]$ is primary since it is a prime ideal. For any $n \geq 1$, the ideal $(x, y)^n$ is primary since it is a power of the maximal ideal (x, y).

(3) The ideal $Q = (x^2, y)$ in the polynomial ring $k[x, y]$ is primary since we have $(x, y)^2 \subseteq (x^2, y) \subseteq (x, y)$. Similarly, $Q' = (4, x)$ in $\mathbb{Z}[x]$ is a $(2, x)$-primary ideal.

(4) Primary ideals need not be powers of prime ideals. For example, the primary ideal Q in the previous example is not the power of a prime ideal, as follows. If $(x^2, y) = P^k$ for some prime ideal P and some $k \geq 1$, then $x^2, y \in P^k \subseteq P$ so $x, y \in P$. Then $P = (x, y)$, and since $y \notin (x, y)^2$, it would follow that $k = 1$ and $Q = (x, y)$. Since $x \notin (x^2, y)$, this is impossible.

(5) If R is Noetherian, and Q is a primary ideal belonging to the prime ideal P, then

$$P^m \subseteq Q \subseteq P$$

for some $m \geq 1$ by Proposition 14. If P is a maximal ideal, then the last statement in Proposition 19 shows that the converse also holds. This is not necessarily true if P