

The following properties of the map \mathcal{I} are very easy exercises. Let A and B be subsets of \mathbb{A}^n .

- (6) If $A \subseteq B$ then $\mathcal{I}(B) \subseteq \mathcal{I}(A)$ (i.e., \mathcal{I} is also *contravariant*).
- (7) $\mathcal{I}(A \cup B) = \mathcal{I}(A) \cap \mathcal{I}(B)$.
- (8) $\mathcal{I}(\emptyset) = k[x_1, \dots, x_n]$ and, if k is infinite, $\mathcal{I}(\mathbb{A}^n) = 0$.

Moreover, there are easily verified relations between the maps \mathcal{Z} and \mathcal{I} :

- (9) If A is any subset of \mathbb{A}^n then $A \subseteq \mathcal{Z}(\mathcal{I}(A))$, and if I is any ideal then $I \subseteq \mathcal{I}(\mathcal{Z}(I))$.
- (10) If $V = \mathcal{Z}(I)$ is an affine algebraic set then $V = \mathcal{Z}(\mathcal{I}(V))$, and if $I = \mathcal{I}(A)$ then $\mathcal{I}(\mathcal{Z}(I)) = I$, i.e., $\mathcal{Z}(\mathcal{I}(\mathcal{Z}(I))) = \mathcal{Z}(I)$ and $\mathcal{I}(\mathcal{Z}(\mathcal{I}(A))) = \mathcal{I}(A)$.

The last relation shows that the maps \mathcal{Z} and \mathcal{I} act as inverses of each other provided one restricts to the collection of affine algebraic sets $V = \mathcal{Z}(I)$ in \mathbb{A}^n and to the set of ideals in $k[\mathbb{A}^n]$ of the form $\mathcal{I}(V)$. In the case where the field k is algebraically closed we shall (in the following two sections) characterize those ideals I that are of the form $\mathcal{I}(V)$ for some affine algebraic set V in terms of purely ring-theoretic properties of the ideal I (this is the famous "Zeros Theorem" of Hilbert, cf. Theorem 32).

Definition. If $V \subseteq \mathbb{A}^n$ is an affine algebraic set the quotient ring $k[\mathbb{A}^n]/\mathcal{I}(V)$ is called the *coordinate ring of V* , and is denoted by $k[V]$.

Note that for $V = \mathbb{A}^n$ and k infinite we have $\mathcal{I}(V) = 0$, so this definition extends the previous terminology. The polynomials in $k[\mathbb{A}^n]$ define k -valued functions on V simply by restricting these functions on \mathbb{A}^n to the subset V . Two such polynomial functions f and g define the *same* function on V if and only if $f - g$ is identically 0 on V , which is to say that $f - g \in \mathcal{I}(V)$. Hence the cosets $\bar{f} = f + \mathcal{I}(V)$ giving the elements of the quotient $k[V]$ are precisely the restrictions to V of ordinary polynomial functions f from \mathbb{A}^n to k (which helps to explain the notation $k[V]$). If x_i denotes the i^{th} coordinate function on \mathbb{A}^n (projecting an n -tuple onto its i^{th} component), then the restriction \bar{x}_i of x_i to V (which also just gives the i^{th} component of the elements in V viewed as a subset of \mathbb{A}^n) is an element of $k[V]$, and $k[V]$ is finitely generated as a k -algebra by $\bar{x}_1, \dots, \bar{x}_n$ (although this need not be a minimal generating set).

Example

If $V = \mathcal{Z}(xy - 1)$ is the hyperbola $y = 1/x$ in \mathbb{R}^2 , then $\mathbb{R}[V] = \mathbb{R}[x, y]/(xy - 1)$. The polynomials $f(x, y) = x$ (the x -coordinate function) and $g(x, y) = x + (xy - 1)$, which are different functions on \mathbb{R}^2 , define the same function on the subset V . On the point $(1/2, 2) \in V$, for example, both give the value $1/2$. In the quotient ring $\mathbb{R}[V]$ we have $\bar{x}\bar{y} = 1$, so $\mathbb{R}[V] \cong \mathbb{R}[x, 1/x]$. For any function $\bar{f} \in \mathbb{R}[V]$ and any $(a, b) \in V$ we have $\bar{f}(a, b) = f(a, 1/a)$ for any polynomial $f \in k[x, y]$ mapping to \bar{f} in the quotient.

Suppose now that $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ are two affine algebraic sets. Since V and W are defined by the vanishing of polynomials, the most natural algebraic maps between V and W are those defined by polynomials:

Definition. A map $\varphi : V \rightarrow W$ is called a *morphism* (or *polynomial map* or *regular map*) of algebraic sets if there are polynomials $\varphi_1, \dots, \varphi_m \in k[x_1, x_2, \dots, x_n]$ such that

$$\varphi((a_1, \dots, a_n)) = (\varphi_1(a_1, \dots, a_n), \dots, \varphi_m(a_1, \dots, a_n))$$

for all $(a_1, \dots, a_n) \in V$. The map $\varphi : V \rightarrow W$ is an *isomorphism* of algebraic sets if there is a morphism $\psi : W \rightarrow V$ with $\varphi \circ \psi = 1_W$ and $\psi \circ \varphi = 1_V$.

Note that in general $\varphi_1, \varphi_2, \dots, \varphi_m$ are not uniquely defined. For example, both $f = x$ and $g = x + (xy - 1)$ in the example above define the same morphism from $V = \mathcal{Z}(xy - 1)$ to $W = \mathbb{A}^1$.

Suppose F is a polynomial in $k[x_1, \dots, x_m]$. Then $F \circ \varphi = F(\varphi_1, \varphi_2, \dots, \varphi_m)$ is a polynomial in $k[x_1, \dots, x_n]$ since $\varphi_1, \varphi_2, \dots, \varphi_m$ are polynomials in x_1, \dots, x_n . If $F \in \mathcal{I}(W)$, then $F \circ \varphi((a_1, a_2, \dots, a_n)) = 0$ for every $(a_1, a_2, \dots, a_n) \in V$ since $\varphi((a_1, a_2, \dots, a_n)) \in W$. Thus $F \circ \varphi \in \mathcal{I}(V)$. It follows that φ induces a well defined map from the quotient ring $k[x_1, \dots, x_m]/\mathcal{I}(W)$ to the quotient ring $k[x_1, \dots, x_n]/\mathcal{I}(V)$:

$$\tilde{\varphi} : k[W] \rightarrow k[V]$$

$$f \mapsto f \circ \varphi$$

where $f \circ \varphi$ is given by $F \circ \varphi + \mathcal{I}(V)$ for any polynomial $F = F(x_1, \dots, x_m)$ with $f = F + \mathcal{I}(W)$. It is easy to check that $\tilde{\varphi}$ is a k -algebra homomorphism (for example, $\tilde{\varphi}(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \tilde{\varphi}(f) + \tilde{\varphi}(g)$ shows that $\tilde{\varphi}$ is additive). Note also the contravariant nature of $\tilde{\varphi}$: the morphism from V to W induces a k -algebra homomorphism from $k[W]$ to $k[V]$.

Suppose conversely that Φ is any k -algebra homomorphism from the coordinate ring $k[W] = k[x_1, \dots, x_m]/\mathcal{I}(W)$ to $k[V] = k[x_1, \dots, x_n]/\mathcal{I}(V)$. Let F_i be a representative in $k[x_1, \dots, x_m]$ for the image under Φ of $\bar{x}_i \in k[W]$ (i.e., $\Phi(x_i \bmod \mathcal{I}(W))$ is $F_i \bmod \mathcal{I}(V)$). Then $\varphi = (F_1, \dots, F_m)$ defines a polynomial map from \mathbb{A}^n to \mathbb{A}^m , and in fact φ is a morphism from V to W . To see this it suffices to check that φ maps a point of V to a point of W since by definition φ is already defined by polynomials. If $g \in \mathcal{I}(W) \subset k[x_1, \dots, x_m]$, then in $k[W]$ we have

$$g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W)) = g(x_1, \dots, x_m) + \mathcal{I}(W) = \mathcal{I}(W) = 0 \in k[W],$$

and so

$$\Phi(g(x_1 + \mathcal{I}(W), \dots, x_m + \mathcal{I}(W))) = 0 \in k[V].$$

Since Φ is a k -algebra homomorphism, it follows that

$$g(\Phi(x_1 + \mathcal{I}(W)), \dots, \Phi(x_m + \mathcal{I}(W))) = 0 \in k[V].$$

By definition, $\Phi(x_i + \mathcal{I}(W)) = F_i \bmod \mathcal{I}(V)$, so

$$g(F_1 \bmod \mathcal{I}(V), \dots, F_m \bmod \mathcal{I}(V)) = 0 \in k[V],$$

i.e.,

$$g(F_1, \dots, F_m) \in \mathcal{I}(V).$$

It follows that $g(F_1(a_1, \dots, a_n), \dots, F_m(a_1, \dots, a_n)) = 0$ for every (a_1, \dots, a_n) in V . This shows that if $(a_1, \dots, a_n) \in V$, then every polynomial in $\mathcal{I}(W)$ vanishes