Definition. A map $\varphi : V \to W$ is called a *morphism* (or *polynomial map* or *regular map*) of algebraic sets if there are polynomials $\varphi_1, \ldots, \varphi_m \in k[x_1, x_2, \ldots, x_n]$ such that

$$\varphi((a_1, \ldots, a_n)) = (\varphi_1(a_1, \ldots, a_n), \ldots, \varphi_m(a_1, \ldots, a_n))$$

for all $(a_1, \ldots, a_n) \in V$. The map $\varphi : V \to W$ is an *isomorphism* of algebraic sets if there is a morphism $\psi : W \to V$ with $\varphi \circ \psi = 1_W$ and $\psi \circ \varphi = 1_V$.

Note that in general $\varphi_1, \varphi_2, \ldots, \varphi_m$ are not uniquely defined. For example, both $f = x$ and $g = x + (xy - 1)$ in the example above define the same morphism from $V = \mathcal{Z}(xy - 1)$ to $W = \mathbb{A}^1$.

Suppose F is a polynomial in $k[x_1, \ldots, x_m]$. Then $F \circ \varphi = F(\varphi_1, \varphi_2, \ldots, \varphi_m)$ is a polynomial in $k[x_1, \ldots, x_n]$ since $\varphi_1, \varphi_2, \ldots, \varphi_m$ are polynomials in $x_1, \ldots, x_n$. If $F \in \mathcal{I}(W)$, then $F \circ \varphi((a_1, a_2, \ldots, a_n)) = 0$ for every $(a_1, a_2, \ldots, a_n) \in V$ since $\varphi((a_1, a_2, \ldots, a_n)) \in W$. Thus $F \circ \varphi \in \mathcal{I}(V)$. It follows that φ induces a well defined map from the quotient ring $k[x_1, \ldots, x_m]/\mathcal{I}(W)$ to the quotient ring $k[x_1, \ldots, x_n]/\mathcal{I}(V)$:

$$\begin{aligned} \widetilde{\varphi} : k[W] &\to k[V] \\ f &\mapsto f \circ \varphi \end{aligned}$$

where $f \circ \varphi$ is given by $F \circ \varphi + \mathcal{I}(V)$ for any polynomial $F = F(x_1, \ldots, x_m)$ with $f = F + \mathcal{I}(W)$. It is easy to check that $\widetilde{\varphi}$ is a k-algebra homomorphism (for example, $\widetilde{\varphi}(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \widetilde{\varphi}(f) + \widetilde{\varphi}(g)$ shows that $\widetilde{\varphi}$ is additive). Note also the contravariant nature of $\widetilde{\varphi}$: the morphism from V to W induces a k-algebra homomorphism from $k[W]$ to $k[V]$.

Suppose conversely that Φ is any k-algebra homomorphism from the coordinate ring $k[W] = k[x_1, \ldots, x_m]/\mathcal{I}(W)$ to $k[V] = k[x_1, \ldots, x_n]/\mathcal{I}(V)$. Let F_i be a representative in $k[x_1, \ldots, x_n]$ for the image under Φ of $\bar{x}_i \in k[W]$ (i.e., $\Phi(x_i \bmod \mathcal{I}(W))$ is $F_i \bmod \mathcal{I}(V)$). Then $\varphi = (F_1, \ldots, F_m)$ defines a polynomial map from $\mathbb{A}^n$ to $\mathbb{A}^m$, and in fact φ is a morphism from V to W. To see this it suffices to check that φ maps a point of V to a point of W since by definition φ is already defined by polynomials. If $g \in \mathcal{I}(W) \subset k[x_1, \ldots, x_m]$, then in $k[W]$ we have

$$g(x_1 + \mathcal{I}(W), \ldots, x_m + \mathcal{I}(W)) = g(x_1, \ldots, x_m) + \mathcal{I}(W) = \mathcal{I}(W) = 0 \in k[W],$$

and so

$$\Phi(g(x_1 + \mathcal{I}(W), \ldots, x_m + \mathcal{I}(W))) = 0 \in k[V].$$

Since Φ is a k-algebra homomorphism, it follows that

$$g(\Phi(x_1 + \mathcal{I}(W)), \ldots, \Phi(x_m + \mathcal{I}(W)) = 0 \in k[V].$$

By definition, $\Phi(x_i + \mathcal{I}(W)) = F_i \bmod \mathcal{I}(V)$, so

$$g(F_1 \bmod \mathcal{I}(V), \ldots, F_m \bmod \mathcal{I}(V)) = 0 \in k[V],$$

i.e.,

$$g(F_1, \ldots, F_m) \in \mathcal{I}(V).$$

It follows that $g(F_1(a_1, \ldots, a_n), \ldots, F_m(a_1, \ldots, a_n)) = 0$ for every $(a_1, \ldots, a_n)$ in V. This shows that if $(a_1, \ldots, a_n) \in V$, then every polynomial in $\mathcal{I}(W)$ vanishes

Proposition 15. The Zariski closure of a subset A in $\mathbb{A}^n$ is $\mathcal{Z}(\mathcal{I}(A))$.

Proof: Certainly $A \subseteq \mathcal{Z}(\mathcal{I}(A))$. Suppose V is any algebraic set containing A: $A \subseteq V$. Then $\mathcal{I}(V) \subseteq \mathcal{I}(A)$ and $\mathcal{Z}(\mathcal{I}(A)) \subseteq \mathcal{Z}(\mathcal{I}(V)) = V$, so $\mathcal{Z}(\mathcal{I}(A))$ is the smallest algebraic set containing A.

If $\varphi : V \to W$ is a morphism of algebraic sets, the image $\varphi(V)$ of V need not be an algebraic subset of W, i.e., need not be Zariski closed in W. For example the projection of the hyperbola $V = \mathcal{Z}(xy - 1)$ in $\mathbb{R}^2$ onto the x-axis has image $\mathbb{R}^1 - \{0\}$, which as we have just seen is not an affine algebraic set.

The next result shows that the Zariski closure of the image of a morphism is determined by the kernel of the associated k-algebra homomorphism.

Proposition 16. Suppose $\varphi : V \to W$ is a morphism of algebraic sets and $\widetilde{\varphi} : k[W] \to k[V]$ is the associated k-algebra homomorphism of coordinate rings. Then

(1) The kernel of $\widetilde{\varphi}$ is $\mathcal{I}(\varphi(V))$.

(2) The Zariski closure of $\varphi(V)$ is the zero set in W of $\ker \widetilde{\varphi}$. In particular, the homomorphism $\widetilde{\varphi}$ is injective if and only if $\varphi(V)$ is Zariski dense in W.

Proof: Since $\widetilde{\varphi} = f \circ \varphi$, we have $\widetilde{\varphi}(f) = 0$ if and only if $(f \circ \varphi)(P) = 0$ for all $P \in V$, i.e., $f(Q) = 0$ for all $Q = \varphi(P) \in \varphi(V)$, which is the statement that $f \in \mathcal{I}(\varphi(V))$, proving the first statement. Since the Zariski closure of $\varphi(V)$ is the zero set of $\mathcal{I}(\varphi(V))$ by the previous proposition, the first statement in (2) follows.

If $\widetilde{\varphi}$ is injective then the Zariski closure of $\varphi(V)$ is $\mathcal{Z}(0) = W$ and so $\varphi(V)$ is Zariski dense. Conversely, suppose $\varphi(V)$ is Zariski dense in W, i.e., $\mathcal{Z}(\mathcal{I}(\varphi(V))) = W$. Then $\mathcal{I}(\varphi(V)) = \mathcal{I}(\mathcal{Z}(\mathcal{I}(\varphi(V)))) = \mathcal{I}(W) = 0$ and so $\ker \widetilde{\varphi} = 0$.

By Proposition 16 the ideal of polynomials defining the Zariski closure of the image of a morphism φ is the kernel of the corresponding k-algebra homomorphism $\widetilde{\varphi}$ in Theorem 6. Proposition 8(1) allows us to compute this kernel using Gröbner bases.

Example: (Implicitization)

A morphism $\varphi : \mathbb{A}^n \to \mathbb{A}^m$ is just a map

$$\varphi((a_1, a_2, \dots, a_n)) = (\varphi_1(a_1, a_2, \dots, a_n), \dots, \varphi_m(a_1, a_2, \dots, a_n))$$

where φ_i is a polynomial. If k is an infinite field, then $\mathcal{I}(\mathbb{A}^m)$ and $\mathcal{I}(\mathbb{A}^n)$ are both 0, so we may write $k[\mathbb{A}^m] = k[y_1, \dots, y_m]$ and $k[\mathbb{A}^n] = k[x_1, \dots, x_n]$. The k-algebra homomorphism $\widetilde{\varphi} : k[\mathbb{A}^m] \to k[\mathbb{A}^n]$ corresponding to φ is then defined by mapping y_i to $\varphi_i = \varphi_i(x_1, \dots, x_n)$. The image $\varphi(\mathbb{A}^n)$ consists of the set of points $(b_1, \dots, b_m)$ with

$$\begin{aligned} b_1 &= \varphi_1(a_1, a_2, \dots, a_n) \\ b_2 &= \varphi_2(a_1, a_2, \dots, a_n) \\ &\ \vdots \\ b_m &= \varphi_m(a_1, a_2, \dots, a_n) \end{aligned}$$

where $a_i \in k$. This is the collection of points in $\mathbb{A}^m$ *parametrized* by the functions $\varphi_1, \dots, \varphi_m$ (with the a_i as parameters). In general such a parametrized collection of points