CHAPTER 18

Representation Theory and Character Theory

18.1 LINEAR ACTIONS AND MODULES OVER GROUP RINGS

For the remainder of the book the groups we consider will be finite groups, unless explicitly mentioned otherwise. Throughout this section F is a field and G is a finite group. We first introduce the basic terminology. Recall that if V is a vector space over F, then $GL(V)$ is the group of nonsingular linear transformations from V to itself (under composition), and if $n \in \mathbb{Z}^+$, then $GL_n(F)$ is the group of invertible $n \times n$ matrices with entries from F (under matrix multiplication).

Definition. Let G be a finite group, let F be a field and let V be a vector space over F.

(1) A *linear representation* of G is any homomorphism from G into $GL(V)$. The *degree* of the representation is the dimension of V.

(2) Let $n \in \mathbb{Z}^+$. A *matrix representation* of G is any homomorphism from G into $GL_n(F)$.

(3) A linear or matrix representation is *faithful* if it is injective.

(4) The *group ring* of G over F is the set of all formal sums of the form

$$\sum_{g\in G} \alpha_g g, \qquad \alpha_g \in F$$

with componentwise addition and multiplication $(\alpha g)(\beta h) = (\alpha\beta)(gh)$ (where α and β are multiplied in F and gh is the product in G) extended to sums via the distributive law (cf. Section 7.2).

Unless we are specifically discussing permutation representations the term "representation" will always mean "linear representation." When we wish to emphasize the field F we shall say F-representation, or representation of G on V over F.

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Recall that if V is a finite dimensional vector space of dimension n, then by fixing a basis of V we obtain an isomorphism $GL(V) \cong GL_n(F)$. In this way any linear representation of G on a finite dimensional vector space gives a matrix representation and vice versa. For the most part our linear representations will be of finite degree and we shall pass freely between linear representations and matrix representations (specifying a basis when we wish to give an explicit correspondence between the two). Furthermore, given a linear representation $\varphi : G \to GL(V)$ of finite degree, a corresponding matrix representation provides numerical invariants (such as the determinant of $\varphi(g)$ for $g \in G$) which are independent of the choice of basis giving the isomorphism between $GL(V)$ and $GL_n(F)$. The exploitation of such invariants will be fundamental to our development.

Before giving examples of representations we recall the group ring FG in greater detail (group rings were introduced in Section 7.2, and some notation and examples were discussed in that section). Suppose the elements of G are $g_1, g_2, \dots, g_n$. Each element of FG is of the form

$$\sum_{i=1}^{n} \alpha_i g_i, \qquad \alpha_i \in F.$$

Two formal sums[1] are equal if and only if all corresponding coefficients of group elements are equal. Addition and multiplication in FG are defined as follows:

$$\sum_{i=1}^{n} \alpha_i g_i \quad + \quad \sum_{i=1}^{n} \beta_i g_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i$$

$$\Big(\sum_{i=1}^{n} \alpha_i g_i\Big)\Big(\sum_{i=1}^{n} \beta_i g_i\Big) = \sum_{k=1}^{n} \Big(\sum_{\substack{i,j \\ g_i g_j = g_k}} \alpha_i \beta_j \Big) g_k$$

where addition and multiplication of the coefficients α_i and β_j is performed in F. Note that by definition of multiplication,

FG is a commutative ring if and only if G is an abelian group.

The group G appears in FG (identifying g_i with $1g_i$) and the field F appears in FG (identifying β with βg_1, where g_1 is the identity of G). Under these identifications

$$\beta\Big(\sum_{i=1}^{n} \alpha_i g_i\Big) = \sum_{i=1}^{n} (\beta \alpha_i) g_i, \qquad \text{for all } \beta \in F.$$

In this way

FG is a vector space over F with the elements of G as a basis.

In particular, FG is a vector space over F of dimension equal to $|G|$. The elements of F commute with all elements of FG, i.e., F is in the *center* of FG. When we wish to emphasize the latter two properties we shall say that FG is an F-*algebra* (in general, an F-algebra is a ring R which contains F in its center, so R is both a ring and an F-vector space).

Note that the operations in FG are similar to those in the F-algebra $F[x]$ (although $F[x]$ is infinite dimensional over F). In some works FG is denoted by $F[G]$, although the latter notation is currently less prevalent.

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[1]The formal sum displayed above is a way of writing the function from G to F which takes the value α_i on the group element g_i. This same "formality" was used in the construction of free modules (see Theorem 6 in Section 10.3).

Examples

(1) If $G = \langle g \rangle$ is cyclic of order $n \in \mathbb{Z}^+$, then the elements of FG are of the form

$$\sum_{i=0}^{n-1} \alpha_i g^i .$$

The map $F[x] \to F\langle g \rangle$ which sends x^k to g^k for all $k \geq 0$ extends by F-linearity to a surjective ring homomorphism with kernel equal to the ideal generated by $x^n - 1$. Thus

$$F\langle g \rangle \cong F[x]/(x^n - 1).$$

This is an isomorphism of F-algebras, i.e., is a ring isomorphism which is F-linear.

(2) Under the notation of the preceding example let $r = 1 + g + g^2 + \cdots + g^{n-1}$, so r is a nonzero element of $F\langle g \rangle$. Note that $rg = g + g^2 + \cdots + g^{n-1} + 1 = r$, hence $r(1 - g) = 0$. Thus the ring $F\langle g \rangle$ contains zero divisors (provided $n > 1$). More generally, if G is any group of order > 1, then for any nonidentity element $g \in G$, $F\langle g \rangle$ is a subring of FG, so FG also contains zero divisors.

(3) Let $G = S_3$ and $F = \mathbb{Q}$. The elements $r = 5(1\,2) - 7(1\,2\,3)$ and $s = -4(1\,2\,3) + 12(1\,3\,2)$ are typical members of $\mathbb{Q}S_3$. Their sum and product are seen to be

$$\begin{aligned} r + s &= 5(1\ 2) - 11(1\ 2\ 3) + 12(1\ 3\ 2) \\ rs &= -20(2\ 3) + 28(1\ 3\ 2) + 60(1\ 3) - 84 \end{aligned}$$

(recall that products (compositions) of permutations are computed from right to left). An explicit example of a sum and product of two elements in the group ring $\mathbb{Q}D_8$ appears in Section 7.2.

Before giving specific examples of representations we discuss the correspondence between representations of G and FG-modules (after which we can simultaneously give examples of both). This discussion closely parallels the treatment of $F[x]$-modules in Section 10.1.

Suppose first that $\varphi : G \to GL(V)$ is a representation of G on the vector space V over F. As above, write $G = \{g_1, \ldots, g_n\}$, so for each $i \in \{1, \ldots, n\}$, $\varphi(g_i)$ is a linear transformation from V to itself. Make V into an FG-module by defining the action of a ring element on an element of V as follows:

$$\Big(\sum_{i=1}^{n} \alpha_i g_i\Big) \cdot v = \sum_{i=1}^{n} \alpha_i \varphi(g_i)(v), \qquad \text{for all } \sum_{i=1}^{n} \alpha_i g_i \in FG,\ v \in V.$$

We verify a special case of axiom 2(b) of a module (see Section 10.1) which shows precisely where the fact that φ is a group homomorphism is needed:

$$\begin{aligned} (g_i g_j) \cdot v &= \varphi(g_i g_j)(v) && \text{(by definition of the action)} \\ &= (\varphi(g_i) \circ \varphi(g_j))(v) && \text{(since } \varphi \text{ is a group homomorphism)} \\ &= \varphi(g_i)(\varphi(g_j)(v)) && \text{(by definition of a composition of linear transformations)} \\ &= g_i \cdot (g_j \cdot v) && \text{(by definition of the action).} \end{aligned}$$

This argument extends by linearity to arbitrary elements of FG to prove that axiom 2(b) of a module holds in general. It is an exercise to check that the remaining module axioms hold.

Note that F is a subring of FG and the action of the field element α on a vector is the same as the action of the ring element $\alpha 1$ on a vector i.e., the FG-module action extends the F action on V.

Suppose now that conversely we are given an FG-module V. We obtain an associated vector space over F and representation of G as follows. Since V is an FG-module, it is an F-module, i.e., it is a vector space over F. Also, for each $g \in G$ we obtain a map from V to V, denoted by $\varphi(g)$, defined by

$$\varphi(g)(v) = g \cdot v \qquad \text{for all } v \in V,$$

where $g \cdot v$ is the given action of the ring element g on the element v of V. Since the elements of F commute with each $g \in G$ it follows by the axioms for a module that for all $v, w \in V$ and all $\alpha, \beta \in F$ we have

$$\begin{aligned} \varphi(g)(\alpha v + \beta w) &= g \cdot (\alpha v + \beta w) \\ &= g \cdot (\alpha v) + g \cdot (\beta w) \\ &= \alpha(g \cdot v) + \beta(g \cdot w) \\ &= \alpha\varphi(g)(v) + \beta\varphi(g)(w), \end{aligned}$$

that is, for each $g \in G$, $\varphi(g)$ is a linear transformation. Furthermore, it follows by axiom 2(b) of a module that

$$\varphi(g_i g_j)(v) = (\varphi(g_i) \circ \varphi(g_j))(v)$$

(this is essentially the calculation above with the steps reversed). This proves that φ is a group homomorphism (in particular, $\varphi(g^{-1}) = \varphi(g)^{-1}$, so every element of G maps to a nonsingular linear transformation, i.e., $\varphi : G \to GL(V)$).

This discussion shows there is a bijection between FG-modules and pairs (V, φ):

$$\left\{ V \text{ an } FG\text{-module} \right\} \longleftrightarrow \left\{ \begin{array}{c} V \text{ a vector space over } F \\ \text{and} \\ \varphi : G \to GL(V) \text{ a representation} \end{array} \right\}.$$

Giving a representation $\varphi : G \to GL(V)$ on a vector space V over F is therefore equivalent to giving an FG-module V. Under this correspondence we shall say that the module V *affords* the representation φ of G.

Recall from Section 10.1 that if a vector space M is made into an $F[x]$-module via the linear transformation T, then the $F[x]$-submodules of M are precisely the T-stable subspaces of M. In the current situation if V is an FG-module affording the representation φ, then a subspace U of V is called *G-invariant* or *G-stable* if $g \cdot u \in U$ for all $g \in G$ and all $u \in U$ (i.e., if $\varphi(g)(u) \in U$ for all $g \in G$ and all $u \in U$). It follows easily that

the FG-submodules of V are precisely the G-stable subspaces of V.

Examples

(1) Let V be a 1-dimensional vector space over F and make V into an FG-module by letting $gv = v$ for all $g \in G$ and $v \in V$. This module affords the representation $\varphi : G \to GL(V)$ defined by $\varphi(g) = I$ = the identity linear transformation, for all $g \in G$. The corresponding matrix representation (with respect to any basis of V) is the homomorphism of G into $GL_1(F)$ which sends every group element to the 1×1 identity matrix. We shall henceforth refer to this as the *trivial representation* of G. The trivial representation has degree 1 and if $|G| > 1$, it is not faithful.

(2) Let $V = FG$ and consider this ring as a left module over itself. Then V affords a representation of G of degree equal to $|G|$. If we take the elements of G as a basis of V, then each $g \in G$ permutes these basis elements under the left regular permutation representation:

$$g \cdot g_i = gg_i.$$

With respect to this basis of V the matrix of the group element g has a 1 in row i and column j if $gg_j = g_i$, and has 0's in all other positions. This (linear or matrix) representation is called the *regular representation* of G. Note that each nonidentity element of G induces a nonidentity permutation on the basis of V so the regular representation is always faithful.

(3) Let $n \in \mathbb{Z}^+$, let $G = S_n$ and let V be an n-dimensional vector space over F with basis $e_1, e_2, \dots, e_n$. Let S_n act on V by defining for each $\sigma \in S_n$

$$\sigma \cdot e_i = e_{\sigma(i)}, \qquad 1 \le i \le n$$

i.e., σ acts by permuting the subscripts of the basis elements. This provides an (injective) homomorphism of S_n into $GL(V)$ (i.e., a faithful representation of S_n of degree n), hence makes V into an FS_n-module. As in the preceding example, the matrix of σ with respect to the basis $e_1, \dots, e_n$ has a 1 in row i and column j if $\sigma \cdot e_j = e_i$ (and has 0 in all other entries). Thus σ has a 1 in row i and column j if $\sigma(j) = i$.

For an example of the ring action, consider the action of FS_3 on the 3-dimensional vector space over F with basis e_1, e_2, e_3. Let σ be the transposition $(1\,2)$, let τ be the 3-cycle $(1\,2\,3)$ and let $r = 2\sigma - 3\tau \in FS_3$. Then

$$\begin{aligned} r \cdot (\alpha e_1 + \beta e_2 + \gamma e_3) &= 2(\alpha e_{\sigma(1)} + \beta e_{\sigma(2)} + \gamma e_{\sigma(3)}) - 3(\alpha e_{\tau(1)} + \beta e_{\tau(2)} + \gamma e_{\tau(3)}) \\ &= 2(\alpha e_2 + \beta e_1 + \gamma e_3) - 3(\alpha e_2 + \beta e_3 + \gamma e_1) \\ &= (2\beta - 3\gamma)e_1 - \alpha e_2 + (2\gamma - 3\beta)e_3. \end{aligned}$$

(4) If $\psi : H \to GL(V)$ is any representation of H and $\varphi : G \to H$ is any group homomorphism, then the composition $\psi \circ \varphi$ is a representation of G. For example, let V be the FS_n-module of dimension n described in the preceding example. If $\pi : G \to S_n$ is any permutation representation of G, the composition of π with the representation above gives a linear representation of G. In other words, V becomes an FG-module under the action

$$g \cdot e_i = e_{\pi(g)(i)}, \qquad \text{for all } g \in G.$$

Note that the regular representation, (2), is just the special case of this where $n = |G|$ and π is the left regular permutation representation of G.

(5) Any homomorphism of G into the multiplicative group $F^\times = GL_1(F)$ is a degree 1 (matrix) representation. For example, suppose $G = \langle g \rangle \cong Z_n$ is the cyclic group of order n and ζ is a fixed n^{th} root of 1 in F. Let $g^i \mapsto \zeta^i$, for all $i \in \mathbb{Z}$. This representation of $\langle g \rangle$ is a faithful representation if and only if ζ is a primitive n^{th} root of 1.